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Contracting Theory with Coincidence of Interest

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Abstract

In the standard models of principal-agent theory, the relationship between the principal and agent is adversarial in the sense that the objective of the agent is to maximize monetary income and minimize effort without regard for the objectives of the principal. In the real world, however, agents often have a personal stake in their contractual responsibilities. Furthermore, standard approaches to optimal contracting involve explicit monetary transfers, thereby excluding from analysis the case of non-profit agents. This paper presents a contracting model—the cooperative model—distinguished from standard models by the following three properties: first, the principal contracts with a non-profit agent for the provision of some commodity; second, the interests of the principal and agent coincide to the extent that their utilities are both increasing in the quality of the commodity provided; third, using a suitable reformulation of the standard moral hazard variable, the optimal contract for the case of quasi-linear preferences has an extremely simple form.

After the cooperative model is formalized, cost-plus and fixed price contracting are defined and compared, the form of the optimal contract is determined for the case of quasi-linear preferences, the suboptimality of cost-plus and fixed price contracting is demonstrated, and the possibility of decentralizing the optimal contract through a menu of linear contracts is explored. Finally, a standard model—the adversarial model—is presented for the purposes of comparison, along with a general model which subsumes both models as special cases. Starting with the adversarial model and altering it in each of the three ways outlined above, it is possible to trace the ramifications of the assumptions underlying the cooperative model.

Contracting Theory with Coincidence of Interest

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The distinguishing mark of principal-agent theory is its recognition of asymmetric information across economic actors. The principal cannot expect an agent to carry out the terms of an unenforceable contract, because the interests of the agent may diverge from those of the principal. This possibility is exploited to its fullest in the standard principal-agent models, which assume that the objective of the agent is to maximize his monetary gains and minimize the effort expended to achieve those gains. While this assumption is useful for revealing the inadequacies of full information economics in a world of asymmetric information, it is clearly an exaggeration of most real world contracting problems. In the labor market, workers are not typically as opportunistic as the agents of principal-agent theory: for example, a typical college professor is anything but an effort minimizer, but rather he has a personal stake in the outcome of his research or the advancement of his students; likewise, an auto mechanic surely has an interest in cars beyond the inducement of his paycheck. There is an abundance of such examples, a consequence of the fact that workers choose their vocation, and they are likely to choose a field which offers them some personal gratification.

In addition to the assumption of divergent interests, the contracts of standard principal-agent models involve the explicit transfer of money to the agent, precluding the possibility that the agent is a non-profit organization. Using a suitable reformulation of the standard moral hazard variable, this paper presents a model in which the principal contracts with a non-profit agent to provide some commodity and in which the interests of the principal and agent coincide to the extent that their utilities are both increasing in the quality of the commodity. Section 1 presents the details of the model with coincidence of interest—the *cooperative model*—largely motivated by a contracting problem of Jet Propulsion Labs (JPL), a NASA center for space exploration under the auspices of Caltech. It is not uncommon for JPL to contract with scientists outside the organization

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for the design and construction of science instruments to be flown on unmanned interplanetary spacecraft. While the scientists certainly care about the quality (or technology level) of the instrument they produce, they may have other interests which lead them to misallocate JPL funds in unobservable ways, resulting in an instrument of lower technical capability. In Section 2, cost-plus and fixed price contracting are defined and conditions are found under which fixed price contracting is preferred by the principal to cost-plus contracting. In Section 3, the optimal contracting problem is posed, the solution is characterized for the case of quasi-linear preferences, and the suboptimality of cost-plus and fixed price contracting is demonstrated. It is seen that the reformulation of the moral hazard variable not only leads to difficulties in the characterization of the incentive compatibility constraint, but also leads to an extremely simple form for the optimal contract. Section 4 explores the possibility of decentralizing the optimal contract through a menu of linear contracts. In Section 5, a standard model—the *adversarial model*—is is adapted from a model of procurement due to Laffont and Tirole [5], and in the context of a general model which subsumes both models as special cases, several modifications of the adversarial model are made in a way which traces the ramifications of the assumptions underlying the cooperative model.

1 The Cooperative Model

The contracting problem is modelled as a four period game of asymmetric information, in which the principal contracts with a non-profit agent for the provision of some good or service. In the first period, the principal offers the agent a menu of contracts, each of which specifies both a level of quality $Q \in \Re_+$ of the commodity and a level of total cost; in the second period, the agent selects from the menu a contract which will in part determine the constraints of his third period maximization problem; in the third period, the agent chooses a level of quality Q and the level of a moral hazard variable $m \in \Re_+$ representing an amount of misallocated funds; and in the fourth period, the principal reimburses the agent for his costs, or if he has observed a breach of contract then the principal costlessly sues the agent, inflicting infinitely large costs.[1] Although only the first and third periods of the game are of analytical interest, the second and fourth are mentioned for the sake of completeness.

The principal's preferences are given by a twice continuously differentiable von Neumann-Morgenstern utility function $U : \Re^2_+ \rightarrow \overline{\Re}$ with extended real values over the quality Q of the commodity and the total amount B to be paid to the agent, which might in turn depend on the values of the agent's choice variables. Furthermore, the principal's utility is assumed to satisfy

$$U_Q > 0, U_B < 0, U_{QQ} < 0, U_{QB} \leq 0, U_{BB} \leq 0, U(0,0) = -\infty,$$

[1] Alternatively, Q can be interpreted as the quantity of some homogeneous good supplied by the agent.

where subscripts denote partial derivatives, as well as the Inada conditions

$$U_Q \overset{Q \to 0}{\longrightarrow} \infty, U_Q \overset{Q \to \infty}{\dashrightarrow} 0, U_B \overset{B \to 0}{\longrightarrow} 0, U_B \overset{B \to \infty}{\longrightarrow} -\infty,$$

where $Q = 0$ represents the state in which the commodity is not provided. The agent's preferences are given by a twice continuously differentiable von Neumann-Morgenstern utility function $V : \Re_+^2 \to \Re$ over the quality Q of the commodity and the level of misallocated funds m. The agent's utility is also assumed to satisfy

$$V_Q > 0, V_m > 0, V_{QQ} < 0, V_{Qm} \geq 0, V_{mm} \leq 0,$$

as well as the Inada condition

$$V_Q \overset{Q \to 0}{\longrightarrow} \infty.$$

In other words, both the principal and the agent are risk averse in quality Q, and they are not risk loving with respect to the budget B and the level of misallocated funds m, respectively; the assumptions on cross partials are made in order to sign certain comparative statics, and are more general than the common assumption of additive separability; the Inada conditions are used to guarantee interior solutions to problems of the principal and agent; and the assumption that $U(0,0) = -\infty$ is used to narrow the scope of viable cost-plus contracts.

The agent's choice of Q and m in the third period is equivalent to the choice of a point on the production frontier determined by the total amount of funding B and the twice continuously differentiable cost function $\mathcal{C} : \Re_+^2 \to \Re$ with functional form $\mathcal{C}(Q, m + \theta) = C(Q) + m + \theta$, where it is assumed that $C : \Re_+ \dashrightarrow \Re$, $C_Q > 0$, $C_{QQ} > 0$, and θ is a random variable with non-atomic distribution function F and density f with support on the interval $\Theta = [\underline{\theta}, \overline{\theta}] \subset \Re_+$. Since the agent is a non-profit organization, it is assumed that the principal will just reimburse the agent for the costs of provision, so that $B = C(Q) + \theta + m$.

The information structure of the game is such that the functions $U, V, \mathcal{C}$, and F are common knowledge to both players. Asymmetric information is introduced by the assumption that the value of θ is known to the agent throughout the game, while the the exact value of θ is unknown to the principal, who only makes observations at the beginning of the fourth period of the quality Q of the commodity and the total cost $\mathcal{C}(Q, m + \theta) = C(Q) + m + \theta$ of the project. That is, the principal can observe $m + \theta = \mathcal{C}(Q, m + \theta) - C(Q)$, but he is unable to distinguish between the values of the two variables m and θ. The value of θ can be interpreted from the view of the principal as a random shock to the fixed cost of the project, which is known to the agent by virtue of his superior technical knowledge, and these possible states of nature can be interpreted as types of agent indexed by the realization of θ.

Since a legally enforceable contract can involve only observable quantities, the principal cannot require that the agent pick $m = 0$.[2] One possible type of legally enforceable contract is a point $(Q, B) \in \Re^2_+$, specifying that the principal will transfer $B = C(Q) + m + \theta$ dollars to the agent, just covering his total cost, in return for which the agent will provide the level of quality Q. It will be useful, however, to consider contracts of the form $(Q, m+\theta) \in \Re^2_+$, which are also enforceable since the principal knows Q and $\mathcal{C}(Q, m+\theta)$, and can then calculate $m+\theta = \mathcal{C}(Q, m+\theta) - C(Q)$. Then the utility to the agent of type $\theta \leq m + \theta'$ from the contract $(Q, m+\theta')$ is $V(Q, m+\theta'-\theta)$, where $m + \theta' - \theta \geq 0$ is the highest level of m which the agent can pick without incurring a breach of contract suit. If, on the other hand, $m + \theta' - \theta < 0$ then $(Q, m + \theta')$ is not feasible for the type θ agent, since the inequality implies that the agent's total cost of providing quality level Q is at least $C(Q) + \theta > C(Q) + m + \theta'$, violating the agent's legally binding cost target. The utility to the principal from the contract $(Q, m + \theta')$ is $U(Q, C(Q) + m + \theta')$.

The literature discerns two types of principal-agent problems stemming from asymmetric information: moral hazard, in which the agent takes an unobservable action, and adverse selection, in which the agent knows the state of the world but the principal does not. In the cooperative model, it seems that the principal faces both moral hazard and adverse selection problems, for the agent not only picks the level of misallocated funds m, which is unobservable to the principal, but he also knows the true value of fixed costs θ, whereas the principal does not. It can be seen, however, that once the adverse selection—or, more appropriately, information revelation—problem is solved, so is the moral hazard problem: once the principal knows the true value of θ he can observe m by calculating $m = \mathcal{C}(Q, m+\theta) - C(Q) - \theta$.

The adverse selection problem is just a mechanism design problem, in which the designer corresponds to the principal, a single participant corresponds to the agent, and the mechanism corresponds to a menu of contracts. As a mechanism design problem, the state space is $\Re^2_+$ with elements $(Q, m+\theta)$, and a direct mechanism is a restriction of the message space to $\Theta = [\underline{\theta}, \overline{\theta}]$, so that a direct mechanism is a mapping $\theta \mapsto (Q(\theta), m(\theta)+\theta)$. That is, by reporting $\hat{\theta}$, the agent selects the contract $(Q(\hat{\theta}), m(\hat{\theta}) + \hat{\theta})$ from the proposed menu of contracts $\{(Q(\theta), m(\theta)+\theta) \in \Re^2_+ | \theta \in \Theta\}$. A truthful revelation direct mechanism is one which satisfies the incentive compatibility constraint: $\forall \theta, \hat{\theta} \in \Theta$

$$V(Q(\theta), m(\theta) + \theta) \geq V(Q(\hat{\theta}), m(\hat{\theta}) + \hat{\theta} - \theta).$$

Invoking the Revelation Principle, the problem of finding an optimal contract for the principal reduces to solving for the function $(Q, m) : \Theta \rightarrow \Re^2_+$ which maximizes his expected utility:

$$\max_{(Q,m)} \int_\Theta U(Q(x), C(Q(x)) + m(x) + x) f(x)\, dx$$

[2] The distinction is sometimes made between observation and verification, which is legally admissible proof of an observation. This will not be maintained here.

subject to incentive compatibility.

2 Cost-plus vs. Fixed Price Contracting

In an ideal world of complete information, *cost-plus* contracting is a promise by the principal to pay for the costs of production $C(Q^c) + \theta$ in exchange for a commodity with quality level Q^c provided by the agent with no misallocated funds. Formally, this is represented by the menu of contracts $\{(Q^c, \theta) \in \Re^2_+ | \theta \in \Theta\}$, where the first coordinate of a contract (Q^c, θ) specifies a quality level Q^c of the commodity to be provided by the agent, the second coordinate specifies the level $m + \theta$ of misallocated funds plus fixed cost permitted by the agent, and the transfer of $C(Q^c) + m + \theta$ from the principal to the agent is implicitly understood. In other words, the principal demands a level of quality and promises to cover total production costs regardless of the realization of fixed cost θ, but not to pay for any misallocated funds $m > 0$ on the part of the agent. With complete information the agent must report his true type, but with incomplete information such a contract is not legally enforceable since a type $\theta < \overline{\theta}$ agent will report a high level $\overline{\theta}$ of fixed cost and misallocate the difference $\overline{\theta} - \theta$.

Since the principal is not able to distinguish between misallocated funds m and the random shock θ, an enforceable cost-plus contract must specify the same amount $m + \theta$ for every type of agent. The cost-plus contract (Q^c, x) then demands that the agent provide the quality level Q^c at cost $C(Q^c) + x$. But the requirements that $m \geq 0$ and $\theta \geq 0$ imply that a type θ agent can accept the contract only if $\theta \leq x$, for even if an agent of type $\theta > x$ chose $m = 0$ there would be no way for him to meet his legally binding cost target. That is, $\theta > x$ implies $C(Q^c) + \theta > C(Q^c) + x$. If $x < \overline{\theta}$ then there is some set of realizations for which the resulting level of quality will be zero, but the condition $U(0, 0) = -\infty$ implies that the expected utility of the principal is infinitely negative whenever any subset of Θ with positive probability measure is excluded from participation.

A menu of cost-plus contracts is then viable for the principal only if it excludes no set of agents with positive probability measure, and in fact, the object of focus in this section is the singleton menu of contracts $\{(Q^c, \overline{\theta})\}$ which excludes no type of agent. In this case, the principal demands the quality level Q^c and reimburses agents of all types $\theta \in \Theta$ for costs up to the point $C(Q) + \overline{\theta}$, beyond which any cost overruns are certainly due to misallocation of funds. Letting $m^c : \Theta \times \Re_+ \to \Re$ denote the level of misallocated funds chosen by an agent of type θ under cost-plus contracting with quality target Q^c, the problem for the agent is

$$\max_m V(Q^c, m)$$

subject to the observational constraint $m \leq \overline{\theta} - \theta$. Noting that the constraint will hold with equality, the solution is simply

$$m^c(\theta) = \overline{\theta} - \theta,$$

where $m^c(\theta)$ is understood to be a function of Q^c. It follows that $\forall\theta \in \Theta\ C(Q) + m + \theta = C(Q) + \overline{\theta}$, so the principal knows with certainty what the total cost, including misallocated funds, will be. Recalling that the principal must exactly reimburse the agent for the cost of the project, the cost-plus budget B^c is given by $B^c = C(Q^c) + \overline{\theta}$, and the principal's problem is then

$$\max_Q U(Q, C(Q) + \overline{\theta}),$$

which under the Inada assumptions has an interior solution Q^c given by the first order condition $U_Q = -U_B C_Q$, and since

$$U_{QQ} + 2U_{QB}C_Q + U_{BB}C_Q + U_B C_{QQ} < 0,$$

it follows that Q^c is unique.

Ideally, *fixed price* contracting is the transfer of a fixed budget B^f from the principal to the agent with no restrictions on the level of misallocated funds chosen by the agent. Under fixed price contracting, the agent has a fixed budget and misallocated funds are diverted from the provision of quality, facing the agent with a trade off. This is not the case for the cost-plus contract $(Q^c, \overline{\theta})$, which binds the principal to cover the cost of the commodity with quality level Q^c regardless of the realization of fixed cost, allowing the agent of type $\theta < \overline{\theta}$ to pick a level of misallocated funds $m^c(\theta) = \overline{\theta} - \theta > 0$ without any effect on the resulting quality level. The advantage of fixed price contracting is that if the preferences of the agent of type θ are close enough to those of the principal then the agent will pick a level of misallocated funds $m^f < m^c$, and if this is true of a big enough subset of Θ then the principal can expect a lower level of misallocated funds under fixed-price contracting than cost-plus and a higher level of quality Q^f.

Letting $Q^f : \Theta \times \Re_+ \rightarrow \Re$ denote the level of quality and $m^f : \Theta \times \Re_+ \rightarrow \Re$ denote the level of misallocated funds chosen by an agent of type θ given a fixed budget B^f, the agent's fixed price problem is

$$\max_{Q,m} V(Q, m)$$

subject to

$$\begin{aligned} C(Q) + m + \theta &\leq B^f \\ Q, m &\geq 0. \end{aligned}$$

Since $U_Q > 0$ and $C_Q > 0$, the first constraint will hold with equality, and the problem becomes

$$\max_Q V(Q, B^f - C(Q) - \theta),$$

subject to

$$B^f - C(Q) - \theta \geq 0,$$

where the constraint $Q \geq 0$ is dropped since $Q > 0$ is guaranteed by the Inada condition on V. The Kuhn-Tucker condition for the agent's fixed price problem is

$$V_Q - C_Q - \lambda C_Q = 0,$$

where the multiplier λ satisfies the complementary slackness conditions

$$\lambda \geq 0,\ B^f - C(Q^f) - \theta \geq 0,\ \lambda(B^f - C(Q^f) - \theta) = 0.$$

Then the solutions to the agent's problem are

$$\begin{aligned} Q^f &= Q^f(\theta, B^f) \\ m^f &= m^f(\theta, B^f) = B - C(Q^f(\theta, B^f)) - \theta. \end{aligned}$$

When $m^f(\theta, B^f) = 0$ it follows that $B^f - C(Q^f) - \theta \equiv 0$, so

$$\begin{aligned} Q^f_\theta &= -\frac{1}{C_Q} < 0 \\ m^f_\theta &= -C_Q Q^f_\theta - 1 = 0, \end{aligned}$$

and when $m^f(\theta, B^f) > 0$ if follows that $\lambda = 0$, so

$$V_Q - V_m C_Q = 0,$$

with the following comparative statics:

$$\begin{aligned} Q^f_\theta &= \frac{V_{Qm} - C_Q V_{mm}}{V_{QQ} - V_{Qm} C_Q - V_m C_{QQ} - C_Q V_{Qm} + C_Q V_{mm}} \\ m^f_\theta &= -C_Q Q^f_\theta - 1. \end{aligned}$$

In particular, $Q^f_\theta \leq 0$ and $m^f_\theta + 1 \geq 0$. For the analysis to follow, the details of the principal's solution for the optimal fixed budget B^f are unnecessary.

Just as cost-plus contracting is represented by a singleton subset of $\Re^2_+$, where one dimension specifies a quality level and the other specifies a level of misallocated funds plus fixed cost, fixed price contracting with budget B^f is represented by the menu of contracts $\{(Q^f(\theta, B^f), m^f(\theta, B^f) + \theta) \in \Re^2_+ | \theta \in \Theta\}$. This set looks something like the one-dimensional path in Figure 1, where the rate of change with respect to θ is given by the directional derivative $(Q^f_\theta, m^f_\theta + 1) = (Q^f_\theta, -C_Q Q^f_\theta)$. The path is pictured sloping downward because $Q^f_\theta < 0$ implies that the slope of the path as a function of $m + \theta$ is $-Q^f_\theta / C_Q Q^f_\theta = -1/C_Q < 0$, while

[Figure 1 about here.]

$Q_\theta^f = 0$ implies that $m^f(\theta, B^f) + \theta$ is constant. Therefore, the path representing the fixed price contract with budget B^f is indeed decreasing.

The results presented in this section attempt to formalize what are intuitively obvious conditions under which the principal will prefer the optimal menu of fixed price contracts to the optimal cost-plus contract. The optimal cost-plus contract $(Q^c, \overline{\theta})$ is compared with a menu of fixed price contracts given by the fixed budget $B^c = C(Q^c) + \overline{\theta}$ in order to find conditions which guarantee that $E[Q^f(\theta, B^c)] > Q^c$. That is, under fixed price contracting the agent is budgeted exactly what would have been spent under the optimal cost-plus contract. When it is the case that $E[Q^f(\theta, B^c)] > Q^c$, there is an additional condition regarding the risk aversion of the principal which ensures that the expected utility of the principal is actually higher with the menu of fixed price contracts $\{(Q^f(\theta, B^c), m^f(\theta, B^c) + \theta) \in \Re^2_+ | \theta \in \Theta\}$ than the optimal cost-plus menu $\{(Q^c, B^c)\}$. Of course, the menu of fixed price contracts with budget B^c is not necessarily the optimal fixed price menu, and in general, the principal will do even better if he allocates the optimal fixed budget B^f.

Before comparing cost-plus and fixed price contracting, however, it is of some interest to consider a menu of restricted fixed price contracts after appending the observational constraint $m + \theta \leq \overline{\theta}$ to a menu of true fixed price contracts. Under a true fixed price contract, the agent is allowed to pick a level of misallocated funds so high that $m^f(\theta, B^f) + \theta > \overline{\theta}$, clearly informing the principal that $m^f(\theta, B^f) > 0$, since he observes total cost $C(Q^f(\theta, B^f)) + m^f(\theta, B^f) + \theta$ and the quality level $Q^f(\theta, B^f)$. But there is no reason why the principal cannot offer the agent a menu of restricted fixed price contracts given by a fixed budget B^r, which the agent is free to spend in any way subject to the constraint that $m^r(\theta, B^r) + \theta \leq \overline{\theta}$, where $m^r(\theta, B^r)$ is the level of misallocated funds chosen by the agent under restricted fixed price contracting. Letting $Q^r(\theta, B^r)$ denote the level of quality chosen by an agent of type θ under the restricted fixed price contract, the following result is immediate.

Proposition 1 $\forall \theta \in \Theta$ $U(Q^r(\theta, B^r), B^r) \geq U(Q^c, B^c)$.

Proof: Let $B^c = C(Q^c) + \overline{\theta}$, noting that this is not necessarily the optimal budget under restricted fixed price contracting. We then have $\forall \theta \in \Theta$

$$C(Q^r(\theta, B^c)) + m^r(\theta, B^c) + \theta = C(Q^c) + \overline{\theta},$$

since agents of all types will spend all of the budget allotted to them, whether on quality or misallocated funds. And since all types of agent must satisfy the observational constraint $m^r(\theta, B^c) + \theta \leq \overline{\theta}$ under the restricted fixed price contract, it follows from the above equation that $\forall \theta \in \Theta$

$$C(Q^r(\theta, B^c)) \geq C(Q^c).$$

Since $C_Q > 0$, we know C is invertible and C^{-1} preserves the above inequality. Therefore, $\forall \theta \in \Theta\ Q^r(\theta, B^c) \geq Q^c$. Then since $U_Q > 0$ it follows that $\forall \theta \in \Theta\ U(Q^r(\theta, B^c), B^c) \geq U(Q^c, B^c)$. □

In comparing the menu of true fixed price contracts to the optimal cost-plus contract, there are four exogenous elements in the cooperative model which are the natural objects of focus: the preferences of the agent, the distribution of the random shock, the cost function, and the preferences of the principal. In what follows of this section, Proposition 2 provides conditions on the agent's preferences which ensure that at least one type θ of agent will pick a level of quality $Q^f(\theta, B^c) > Q^c$, while Propositions 3 to 7 offer conditions under which the expected quality level under fixed price contracting with budget B^c is at least as high as the level of quality Q^c under the optimal cost-plus contract. The final proposition of the section, Proposition 8, gives a sufficient condition on the utility function of the principal for a higher expected level of quality to result in a higher level of expected utility. The next two propositions, concerning the preferences of the agent, express the idea discussed above that when the agent cares enough about the quality level of the commodity, fixed price contracting does better for the principal. The rest of the propositions of this section consider the fixed price contract with budget B^c, and unless otherwise stated, $Q^f(\theta)$ will denote $Q^f(\theta, B^c)$ and $m^f(\theta)$ will denote $m^f(\theta, B^c)$.

Proposition 2 *If $V_m(Q^c, B^c - C(Q^c) - \underline{\theta})C_Q(Q^c)$ is low enough, or $V_Q(Q^c, B^c - C(Q^c) - \underline{\theta})$ is high enough, then $Q^f(\underline{\theta}) > Q^c$.*

Proof: As shown in Figure 2, this follows from the first order conditions $V_Q = V_m C_Q$ of the agent's problem under fixed price contracting, and the observation that the term $B^c - C(Q^c) - \underline{\theta}$ is just the level of misallocated funds for the agent of type $\underline{\theta}$ when the fixed budget is B^c and he picks the level of technology Q^c.

[Figure 2 about here.]

Note that under the assumptions on V and C,

$$\begin{aligned} \frac{d}{dQ} V_m(Q, B^c - C(Q) - \underline{\theta})C_Q(Q) &= (V_{Qm} - V_{mm}C_Q)C_Q + V_m C_{QQ} > 0 \\ \frac{d}{dQ} V_Q(Q, B^c - C(Q) - \underline{\theta}) &= V_{QQ} - V_{Qm}C_Q < 0, \end{aligned}$$

which is why the first function is drawn increasing and the second decreasing. □

Proposition 3 *If $V_m(Q^c, 0)C_Q(Q^c)$ is low enough, or if $V_Q(Q^c, 0)$ is high enough, then $E[Q^f] > Q^c$.*

Proof: Since $Q^f_\theta \leq 0$, it suffices to show $Q^f(\overline{\theta}) > Q^c$, and again using the first order conditions of the agent's problem, this is just what the antecedent of proposition entails. □

The following three propositions state conditions on the distribution F of θ which will guarantee that $E[Q^f] > Q^c$. The first of the three is a crude formalization—using a Taylor series approximation—of the intuition that when the expected value of θ is small then the expected value of Q^f should be big. In other words, since Q^f is non-increasing in θ, a smaller $E[\theta]$ should correspond to a larger $E[Q^f]$. In Propositions 4 and 5, let $\tilde{\theta}$ be defined not necessarily uniquely by $Q^f(\tilde{\theta}) \equiv Q^c$, and for Proposition 6, let $b = \sup\{-Q^f_\theta(\theta) | \theta \in \Theta\}$.

Proposition 4 *If $E[\theta] < \tilde{\theta}$ then $E[Q^f]$ is approximately greater than or equal to Q^c. Furthermore, $E[Q^f]$ is higher when $E[\theta] - \tilde{\theta}$ is lower.*

Proof: Consider the first order Taylor series expansion of $Q^f(\theta)$ around $\tilde{\theta}$:

$$Q^f(\theta) = Q^f(\tilde{\theta}) + Q^f_\theta(\tilde{\theta})(\theta - \tilde{\theta}) + R(\theta),$$

where $R(\theta)$ is the remainder. Since this is true for all types of agent, we can take expectations, noting that $Q^f(\tilde{\theta}) = Q^c$:

$$\begin{aligned} E[Q^f] &= Q^c + (E[\theta] - \tilde{\theta})Q^f_\theta(\tilde{\theta}) + E[R(\theta)] \\ &\approx Q^c + (E[\theta] - \tilde{\theta})Q^f_\theta(\tilde{\theta}) \\ &\geq Q^c, \end{aligned}$$

where the last inequality follows since $Q^f_\theta \leq 0$ and it is assumed that $E[\theta] - \tilde{\theta} < 0$. □

Proposition 5 *Suppose $Q^f(\underline{\theta}) > Q^c$. If $F(\tilde{\theta})$ is close enough to one then $E[Q^f] > Q^c$.*

Proof: Note that

$$\begin{aligned} E[Q^f] &= \int_{\underline{\theta}}^{\tilde{\theta}} Q^f(x)f(x)\,dx + \int_{\tilde{\theta}}^{\overline{\theta}} Q^f(x)f(x)\,dx \\ &\geq Q^c F(\tilde{\theta}) + \int_{\underline{\theta}}^{\tilde{\theta}} (Q^f(x) - Q^c)f(x)\,dx + Q^f(\overline{\theta})(1 - F(\tilde{\theta})). \end{aligned}$$

It then follows that $E[Q^f] > Q^c$ if

$$Q^c F(\tilde{\theta}) + \int_{\underline{\theta}}^{\tilde{\theta}} (Q^f(x) - Q^c) f(x)\, dx + Q^f(\overline{\theta})(1 - F(\hat{\theta})) > Q^c,$$

which is equivalent to the condition

$$\frac{\int_{\underline{\theta}}^{\tilde{\theta}} (Q^f(x) - Q^c) f(x)\, dx}{1 - F(\tilde{\theta})} + Q^f(\overline{\theta}) > Q^c. \tag{1}$$

To see that the numerator is positive, note that the continuity of Q^f and the fact that $Q^f(\underline{\theta}) > Q^c$ imply $\exists \delta > 0$ such that $\theta \leq \underline{\theta} + \delta$ implies $Q^f(\theta) > Q^c$. Moreover, since Q^f is non-increasing, $\underline{\theta} + \delta < \tilde{\theta}$. Then

$$\begin{aligned} \int_{\underline{\theta}}^{\tilde{\theta}} (Q^f(x) - Q^c) f(x)\, dx &= \int_{\underline{\theta}}^{\underline{\theta}+\delta} (Q^f(x) - Q^c) f(x)\, dx \\ &+ \int_{\underline{\theta}+\delta}^{\tilde{\theta}} (Q^f(x) - Q^c) f(x)\, dx \geq \int_{\underline{\theta}}^{\underline{\theta}+\delta} (Q^f(x) - Q^c) f(x)\, dx > 0. \end{aligned}$$

Then since the limit of the left hand term in (1) goes to infinity as $F(\tilde{\theta})$ goes to one, the proposition follows. □

Proposition 6 *If* $0 \leq b(E[\theta] - \underline{\theta}) < Q^f(\underline{\theta}) - Q^c$ *then* $E[Q^f] > Q^c$.

Proof: Note that

$$\begin{aligned} E[Q^f] &= \int_{\underline{\theta}}^{\overline{\theta}} Q^f(x) f(x)\, dx = \int_{\underline{\theta}}^{\overline{\theta}} \left(Q^f(\underline{\theta}) + \int_{\underline{\theta}}^{x} Q^f_\theta(y)\, dy \right) f(x)\, dx \\ &= Q^f(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\underline{\theta}}^{x} Q^f_\theta(y)\, dy \right) f(x)\, dx \\ &\geq Q^f(\underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\underline{\theta}}^{x} b\, dy \right) f(x)\, dx \\ &= Q^f(\underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} b(x - \underline{\theta}) f(x)\, dx \\ &= Q^f(\underline{\theta}) - b(E[\theta] - \underline{\theta}) \\ &> Q^f(\underline{\theta}) - Q^f(\underline{\theta}) + Q^c = Q^c. \square \end{aligned}$$

As depicted in Figure 3, Proposition 6 simply says that if Q^f is bounded below by a function $Q^f(\underline{\theta}) - b(\theta - \underline{\theta})$ and the expected value of that function is greater than Q^c then so is $E[Q^f]$.

[Figure 3 about here.]

Since $m^f(\theta)+\theta < \overline{\theta}$ implies $Q^f(\theta) > Q^c$, it is natural to expect that $E[m^f(\theta)+\theta] < \overline{\theta}$ implies $E[Q^f] > Q^c$. As the following proposition shows, however, this is not true unless the cost function is not too convex. That is, the conjecture is true only when the risk premium $\gamma > 0$, defined by $C(E[Q^f]+\gamma) \equiv E[C(Q^f)]$ is small enough. Also, let $\alpha = C^{-1}(C(Q^c)+\overline{\theta} - E[m^f(\theta)+\theta]) - Q^c$.

Proposition 7 *$E[Q^f] > Q^c$ if and only if $E[m^f(\theta)+\theta] < \overline{\theta}$ and $\gamma < \alpha$.*

Proof: First, consider the "if" direction. Note that $E[m^f(\theta)+\theta] < \overline{\theta}$ implies that $\alpha > 0$, so the conditional is not vacuously true. We know $\forall\theta \in \Theta$

$$C(Q^f(\theta)) + m^f(\theta) + \theta = C(Q^c) + \overline{\theta},$$

which implies

$$E[C(Q^f)] = C(Q^c) + \overline{\theta} - E[m^f(\theta)+\theta]$$

$\Longleftrightarrow$

$$C(E[Q^f]+\gamma) = C(Q^c) + \overline{\theta} - E[m^f(\theta)+\theta]$$

$\Longleftrightarrow$

$$\begin{aligned} E[Q^f] &= C^{-1}(C(Q^c) + \overline{\theta} - E[m^f(\theta)+\theta]) - \gamma \\ &= \alpha + Q^c - \gamma > Q^c, \end{aligned}$$

which is the desired result. The "only if" direction follows easily. □

The final proposition of the section formalizes the intuition that when the principal is not too risk averse, a higher expected level of quality under fixed price contracting *with budget* B^c than under the optimal cost-plus contract yields the principal a higher level of expected utility. As discussed above, when this condition is fulfilled the expected utility of the principal under the *optimal* menu of fixed price contracts is also higher than his utility under the optimal cost-plus contract. Define the principal's risk premium $\pi > 0$ by $U(E[Q^f]-\pi, B^c) \equiv E[U(Q^f, B^c)]$.

Proposition 8 *If π is low enough and $E[Q^f] > Q^c$ then $E[U(Q^f, B^c)] > U(Q^c, B^c)$.*

Proof: Suppose that $0 \leq \pi < E[Q^f] - Q^c$. Then

$$\begin{aligned} U(Q^c, B^c) &= U(Q^c + \pi - \pi, B^c) \\ &< U(Q^c + E[Q^f] - Q^c - \pi, B^c) \\ &= U(E[Q^f] - \pi, B^c) = E[U(Q^f, B^c)]. \square \end{aligned}$$

3 The Optimal Mechanism

Now that conditions have been established under which fixed price contracting is preferred by the principal to cost-plus contracting, this section will explore the optimal contracting problem. Applying the techniques of control theory to find the optimal incentive compatible mechanism, the first half of the problem is finding a manageable form of the incentive compatibility constraint. This is the subject of Propositions 11 and 12, along with Corollary 13. In Propositions 14 and 15 the form of the optimal incentive compatible mechanism is solved for in the special case of quasi-linear preferences of the principal and agent, while Corollaries 16 and 17 show the suboptimality of cost-plus and fixed price contracting.

If there were no asymmetric information in the cooperative model, so that the principal observed the agent's type, the optimal mechanism would correspond to the menu of contracts $\{(Q^*(\theta), \theta) \in \Re^2_+ | \theta \in \Theta\}$, specifying a level of quality and a level of misallocated funds plus fixed cost, where $Q^*(\theta)$ is the first best level of technology as a function of the random shock. That is, $Q^*(\theta)$ is the solution to

$$\max_Q U(Q, C(Q) + \theta),$$

which by the assumptions on the principal's utility function is uniquely characterized by the necessary first order condition

$$U_Q + U_B C_Q = 0.$$

The comparative statics of the problem yield

$$Q^*_\theta = \frac{-U_{QB} - C_Q U_{BB}}{U_{QQ} + 2U_{QB}C_Q + U_B C_{QQ} + C_Q U_{QB} C_Q} \leq 0.$$

Of course, the real world is not so simple, and the principal must confront the problem of finding the optimal menu of contracts with asymmetric information.

Invoking the Revelation Principle, the optimal contracting problem is greatly simplified by restricting attention to incentive compatible direct mechanisms $\theta \mapsto (Q(\theta), m(\theta) + \theta)$ with $\forall \theta, \hat{\theta} \in \Theta$

$$V(Q(\theta), m(\theta) + \theta) \geq V(Q(\hat{\theta}), m(\hat{\theta}) + \hat{\theta} - \theta).$$

Furthermore, the principal's feasible set is restricted to mechanisms which are piecewise continuously differentiable, with two additional qualifications: Q_θ and m_θ are bounded on Θ and $m_\theta + 1$ does not fluctuate too much around zero.[3] Such a mechanism can be

[3]A function is piecewise continuously differentiable if it has a continuous derivative at all but a finite number of points, and at those points the function still has left and right derivatives.

To see that a piecewise continuously differentiable function is not necessarily absolutely continuous—

represented by a graph with Q on the vertical axis and $m + \theta$ on the horizontal axis. The indifference curves of agent types can be overlayed on the same graph by translating them to the right by the amount θ, since for any contract $(Q(\hat{\theta}), m(\hat{\theta}) + \hat{\theta})$ the resulting level of funds misallocated by an agent of type θ is just $m(\hat{\theta}) + \hat{\theta} - \theta$. Note that when agents of all types choose $m = 0$, the horizontal axis then also represents types θ of agents, so the function $Q^* : \Theta \longrightarrow \Re$ can be represented on this graph, as in Figure 4.

[Figure 4 about here.]

This graphical insight is exemplified in the following proposition.

Proposition 9 *If $Q^*(\theta)$ is concave then the principal can achieve the first best if $\forall\theta \in \Theta$*

$$-\frac{V_m(Q^*(\theta), 0)}{V_Q(Q^*(\theta), 0)} \geq Q^*_\theta(\theta).$$

Proof: Consider the mechanism $\theta \mapsto (Q^*(\theta), \theta)$, specifying the first best level of quality and a level of $m + \theta$ for each agent type θ. This implies that $m = 0$ for agents of all types. The proposition then follows from inspection of Figure 4, and the fact that, under the assumptions in Section 1, the indifference contours of all agent types are convex as functions of m:

$$\frac{d}{dm}\left(-\frac{V_m}{V_Q}\right) = -\frac{V_Q V_{mm} - V_m V_{Qm}}{(V_Q)^2} \geq 0.$$

□

The proposition implicitly uses the fact that a type θ_2 agent cannot imitate a type $\theta_1 < \theta_2$ agent, for if the type θ_2 agent were to report θ_1, there would be no way for him to produce the quality level $Q^*(\theta_1)$ at his legally binding cost target $C(Q^*(\theta_1)) + \theta_1$. That is, even if the type θ_2 agent picks $m = 0$, his cost will be $C(Q^*(\theta_1)) + \theta_2 > C(Q^*(\theta_1)) + \theta_1$. The proposition follows since the convexity of the agents' indifference curves ensures that

---

so that it cannot necessarily be written as the integral of its derivative—consider the function $f(x) = x^2 \sin(1/x^2)$ on the interval $[0, \sqrt{\pi/2}]$. Note that f is differentiable everywhere on $(0, \pi/2)$, with one-sided derivatives at 0 and $\pi/2$, and f' is discontinuous only at zero, so f is piecewise continuously differentiable. Furthermore, the variation of f is $V(f) = \sum_{n=1}^{\infty} 2/n\pi = \infty$, and therefore f is not absolutely continuous. The qualification that Q_θ and m_θ are bounded, along with the piecewise continuous differentiability of Q and m, is sufficient for the absolute continuity of Q and m.

The second qualification is needed for the proof of Proposition 11. Letting Λ_c denote the set of all types θ for which $m_\theta(\theta) + 1 = 0$ and $m(\theta) + \theta = c$, it is required that the set $\{c \in \Re_+ | \Lambda_c \neq \emptyset\}$ be at most countably infinite. Since $m_\theta + 1 = 0$ implies that $m(\theta) + \theta$ is constant, the restriction implies that there is at most a countably infinite number of points c at which two or more agent types θ_1 and θ_2 are pooled, in the sense that $m(\theta_1) + \theta_1 = m(\theta_2) + \theta_2 = c$.

it will never be in the interest of a type θ_1 agent to imitate a type $\theta_2 > \theta_1$ agent. This observation is generalized in Lemma 1, according to which a type θ_2 agent can successfully imitate a type $\theta_1 < \theta_2$ agent if and only if $m(\theta_1) + \theta_1 \geq \theta_2$. It follows that a type θ_1 agent can always successfully imitate a type $\theta_2 > \theta_1$ agent. This leads to the following characterization of incentive compatibility within the context of the cooperative model.

Proposition 10 *The mechanism $\theta \mapsto (Q(\theta), m(\theta) + \theta)$ is incentive compatible if and only if $\forall \theta, \hat{\theta} \in \Theta$ such that $m(\hat{\theta}) + \hat{\theta} \geq \theta$*

$$V(Q(\theta), m(\theta)) \geq V(Q(\hat{\theta}), m(\hat{\theta}) + \hat{\theta} - \theta).$$

Proof: Incentive compatibility obviously implies this condition. Its sufficiency follows from the fact that this is just the definition of incentive compatibility except that it ignores the possibility of a type θ_2 agent imitating a type θ_1 agent when $m(\theta_1) + \theta_1 < \theta_2$. That this omission is appropriate follows from Lemma 1. □

That is, when facing the direct mechanism $\theta \mapsto (Q(\theta), m(\theta) + \theta)$, the type θ agent reports as his type the solution to the problem

$$\max_{\hat{\theta}} V(Q(\hat{\theta}), m(\hat{\theta}) + \hat{\theta} - \theta)$$

subject to

$$\theta \leq m(\hat{\theta}) + \hat{\theta}.$$

The Kuhn-Tucker condition for the agent's problem is

$$V_Q Q_\theta + V_m(m_\theta + 1) + \mu(m_\theta + 1) = 0,$$

where μ is the Kuhn-Tucker multiplier satisfying the complementary slackness conditions

$$\mu \geq 0,\ \theta \leq m(\hat{\theta}) + \hat{\theta},\ \mu(m(\hat{\theta}) + \hat{\theta} - \theta) = 0.$$

The solution to the agent's problem is not necessarily interior with $m > 0$, so the first order conditions for an interior maximum are not necessary for incentive compatibility. For example, Proposition 9 shows that under certain conditions the first best mechanism is incentive compatible with $\forall \theta \in \Theta\ m(\theta) = 0$, so that no type θ_2 agent can imitate a type $\theta_1 < \theta_2$ agent. That is, no agent of any type can falsify his reported type downward, so the first order conditions for an interior maximum need not hold for *any* type of agent. Although this shows that the first and second order conditions for an interior maximum of the agent's problem do not characterize incentive compatibility in the cooperative model, other necessary and sufficient conditions are derived in Propositions 11, 12, and Corollary 13.

In order to go on to characterize incentive compatibility, it will be useful to consider the function $\mathcal{Q} : M \to \Re$ which maps a level of misallocated funds plus fixed costs $m(\theta) + \theta \in M = [m(\underline{\theta}) + \underline{\theta}, m(\overline{\theta}) + \overline{\theta}]$ to a corresponding level of quality $Q(\theta)$. Lemmata 2 and 3 show that under the assumptions of Proposition 11 $\mathcal{Q}$ is in fact a function, so that it maps a point $x \in M$ to a single point $\mathcal{Q}(x) \in \Re_+$, and that it is non-increasing. The former would be obviously true if $m(\theta) + \theta$ were invertible, but this is not the case when $m_\theta + 1 = 0$. Note that if $m_\theta(\theta) + 1 \neq 0$ then $\mathcal{Q}_x(m(\theta) + \theta) = Q_\theta(\theta)/(m_\theta(\theta) + 1)$. It will also be useful to distinguish between pooled and non-pooled sets: let $\Lambda_c = \{\theta \in \Theta | m_\theta(\theta) + 1 = 0 \text{ and } m(\theta) + \theta = c\}$, so that Λ_c is *pooled* at c if $\Lambda_c \neq \emptyset$. Define a point $c \in \Re_+$ to be a *pooling point* if Λ_c is a pooled set. Finally, let $P_\Lambda = \{c \in \Re_+ | \Lambda_c \neq \emptyset\}$ be the set of all pooling points, and note that the restrictions on m specified in footnote 3 entail that P_Λ is at most countably infinite.

Proposition 11 *When the agent has quasi-linear utility $V(Q, m) = V(Q) + m$, the following conditions are sufficient for incentive compatibility:* $\forall \theta \in \Theta$

I $Q_\theta \leq 0$

II $m_\theta + 1 \geq 0$

III $V_Q Q_\theta + m_\theta + 1 \leq 0$

IV $m(\theta)(V_Q Q_\theta + m_\theta + 1) = 0.$

Proof: (Appendix A) It is supposed, contrary to the proposition, that there are two agents, one of whom θ' is better off reporting the other's type θ'' and can avoid detection by the principal. The proof considers two cases: $\theta' < \theta''$ and $\theta'' < \theta'$. The second case admits a standard sufficiency proof, for the reason that the first order condition $V_Q Q_\theta + m_\theta + 1 = 0$ must hold for all agent types between θ'' and θ'.

The first case, however, requires some innovation. Roughly, it is shown that there is some type $\tilde{\theta}$ agent between agents θ' and θ'' such that $m(\tilde{\theta}) + \tilde{\theta}$ is not a pooling point and the graph of $\mathcal{Q}$ is flatter than the agent's indifference curve at the point $(Q(\tilde{\theta}), m(\tilde{\theta}) + \tilde{\theta})$. That is,

$$-\frac{1}{V_Q(Q(\tilde{\theta}))} < \mathcal{Q}_x(m(\tilde{\theta}) + \tilde{\theta}) = \frac{Q_\theta(\tilde{\theta})}{m_\theta(\tilde{\theta}) + 1},$$

or equivalently,

$$V_Q(Q(\tilde{\theta}))Q_\theta(\tilde{\theta}) + m_\theta(\tilde{\theta}) + 1 > 0,$$

contradicting (**III**).

Proposition 12 *If the utility functions of the principal and agent are quasi-linear*

$$\begin{aligned} U(Q,B) &= U(Q) - B \\ V(Q,m) &= V(Q) + m \end{aligned}$$

and the mapping $\theta \mapsto (Q(\theta), m(\theta) + \theta)$ *is an optimal incentive compatible mechanism then the following conditions hold:* $\forall \theta \in \Theta$

I $Q_\theta \leq 0$

II $V_Q Q_\theta + m_\theta + 1 = 0.$

Proof: (Appendix A) First it is supposed, contrary to the proposition, that there is an optimal incentive compatible mechanism $(Q(\theta), m(\theta) + \theta)$ which violates (**I**), so that $\exists \theta \in \Theta$ such that $Q_\theta(\theta) > 0$. It is shown that this must be true for some open set $B \subset \Theta$ the image of which under Q lies either entirely above or entirely below the value $\overline{Q}$, defined uniquely by $U_Q(\overline{Q}) + V_Q(\overline{Q}) \equiv C_Q(\overline{Q})$. In particular it must be true for some interval $[\theta_1, \theta_2] \subset B$. Then it is shown that the principal does strictly better with a new mechanism $(Q', m' + \theta)$ which assigns—depending on where the image of B lies—to each agent type in $[\theta_1, \theta_2]$ the contract $(Q(\theta_1), m(\theta_1) + \theta_1)$ or $(Q(\theta_2), m(\theta_2) + \theta_2)$. Then the original mechanism could not have been optimal.

The proof of (**II**) supposes that there is an optimal incentive compatible mechanism $(Q(\theta), m(\theta) + \theta)$ which violates the condition. It is easily shown that incentive compatibility implies $V_Q Q_\theta + m_\theta + 1 \leq 0$, so the supposition reduces to $\exists \theta' \in \Theta$ such that $V_Q(Q(\theta'))Q_\theta(\theta') + m_\theta(\theta') + 1 < 0$, or in other words, the menu of contracts $\{(Q(\theta), m(\theta) + \theta) \in \Re^2_+ | \theta \in \Theta\}$ is flatter at θ' than the agent's indifference curve. Again, this must be true for some open set $B \subset \Theta$ the image of which under Q lies either entirely above or entirely below the value $Q^*(\theta) = Q^*$. Once this is done, it is shown that a 'variation' ν can be added or subtracted—depending on where the image of B lies—to the function $Q(\theta)$ to yield a new mechanism $(Q \pm \nu, m + \theta)$ which does strictly better for the principal. Then the original mechanism could not have been optimal.

It follows from the previous result that if the utility functions of the principal and agent are quasi-linear then an optimal mechanism $\theta \mapsto (Q(\theta), m(\theta) + \theta)$ is incentive compatible only if $Q_\theta \leq 0$ and $V_Q Q_\theta + m_\theta + 1 = 0$. The next result shows that these conditions are in fact sufficient for an optimal contract to be incentive compatible.

Corollary 13 *If the utility functions of the principal and agent are quasi-linear then an optimal mechanism* $\theta \mapsto (Q(\theta), m(\theta) + \theta)$ *is incentive compatible if and only if the* *following two conditions hold:* $\forall\theta \in \Theta$

$$\begin{aligned} Q_\theta &\leq 0 \\ V_Q Q_\theta + m_\theta + 1 &= 0. \end{aligned}$$

Proof: It was shown in Proposition 12 that these conditions are necessary for incentive compatibility, and from Proposition 11 it follows that they are sufficient if they imply $m_\theta + 1 \geq 0$. From the assumptions of the proposition we have

$$m_\theta + 1 = -V_Q Q_\theta \geq 0.$$

where we use the assumption that $V_Q > 0$. □

It is now possible to set up the optimal contracting problem as a control theory problem with the necessary and sufficient conditions for incentive compatibility in Corollary 13 as constraints on the instruments Q and m. These conditions are not necessary for incentive compatibility in general, and it may seem that they rule out too many mechanisms, possibly leaving only inefficient mechanisms as solutions to the problem. But these conditions are necessary for *all* optimal incentive compatible mechanisms, and consequently they do not rule out any such mechanism. Furthermore, a non-negativity constraint on m must be appended to the problem, for otherwise the solution for m in the control theory problem would be infinitely negative.

The state variable of the problem will be $\mathcal{V}(\theta) = V(Q(\theta)) + m(\theta)$, which represents the utility to the agent of type θ when he reports his true type, and the control variable of the problem will be Q. Note that Q and $\mathcal{V}$ implicitly define the function $m^*(Q, \mathcal{V})$, which gives the level of misallocated funds m as a function of the level of quality Q and the type θ agent's truthful utility $\mathcal{V}$. The partials of m^* are

$$\begin{aligned} m^*_Q &= -V_Q \\ m^*_{\mathcal{V}} &= 1. \end{aligned}$$

Note also that

$$\mathcal{V}_\theta = V_Q Q_\theta + m_\theta,$$

so the incentive compatibility constraint $V_Q Q_\theta + m_\theta + 1 = 0$ reduces to $\mathcal{V}_\theta = -1$. The constraint $Q_\theta \leq 0$ will be dropped, but it will be seen that the solution to the relaxed problem in fact satisfies the omitted condition.

The control theory problem is then

$$\max_Q \int_{\underline{\theta}}^{\overline{\theta}} [U(Q) - C(Q) - m^*(Q, \mathcal{V}) - x] f(x)\, dx$$

subject to

$$
\begin{aligned}
\mathcal{V}_\theta &= -1 && (2)\\
m^*(Q,\mathcal{V}) &\geq 0, && (3)
\end{aligned}
$$

with both endpoints free. The Hamiltonian of the problem is

$$
H = [U(Q) - C(Q) - m^*(Q,\mathcal{V}) - x]f(x) - \tau + \gamma m^*,
$$

where τ is the multiplier for (2) and γ is the multiplier for (3), satisfying the complementary slackness conditions

$$
\gamma \geq 0,\ \gamma m^* = 0,\ m^* \geq 0. \tag{4}
$$

The maximum principle implies that a solution to the control problem must satisfy

$$
\begin{aligned}
(U_Q - C_Q + V_Q)f - \gamma V_Q &= 0 && (5)\\
\tau_\theta &= f - \gamma,
\end{aligned}
$$

and the transversality conditions are $\tau(\underline{\theta}) = \tau(\overline{\theta}) = 0$.

Lemma 6 shows that these necessary conditions imply the omitted condition $Q_\theta \leq 0$, so that any solution of the relaxed problem is optimal and incentive compatible, and the next proposition shows that the above necessary conditions actually determine the form of the optimal incentive compatible mechanism. Let $\overline{Q}$ be uniquely defined by

$$
U_Q(\overline{Q}) + V_Q(\overline{Q}) - C_Q(\overline{Q}) \equiv 0,
$$

and let θ^* be uniquely defined by

$$
V^{-1}(\overline{\theta} + V(Q(\overline{\theta})) - \theta^*) \equiv \overline{Q}.
$$

Proposition 14 *If the utility functions of the principal and agent are quasi-linear, the optimal incentive compatible mechanism $\theta \mapsto (Q(\theta), m(\theta)+\theta)$ is given by*

$$
(Q(\theta), m(\theta)+\theta) = \begin{cases} (V^{-1}(V(Q(\overline{\theta})) + \overline{\theta} - \theta), \theta) & \text{if } \theta \geq \theta^* \\ (\overline{Q}, \theta^*) & \text{if } \theta \leq \theta^*. \end{cases}
$$

Proof: (Appendix A) The result follows from Lemmata 4 through 6.

According to Proposition 14, for the optimal incentive compatible mechanism the contracts of agent types $\theta \geq \theta^*$ correspond to points on the agent's indifference curve $\overline{V}$ through the type $\overline{\theta}$ agent's contract $(Q(\overline{\theta}), \overline{\theta})$. This is graphed in Figure 5.

[Figure 5 about here.]

To see that this is the case, note that we can take $\overline{V}$ as the image of $[Q(\overline{\theta}), Q(\underline{\theta})]$ under a continuously differentiable function into $M = [m(\underline{\theta}) + \underline{\theta}, m(\overline{\theta}) + \overline{\theta}]$ with slope $-V_Q(Q)$. Then we have $\forall \theta > \theta^*$

$$\begin{aligned} m(\theta) + \theta &= m(\overline{\theta}) + \overline{\theta} + \int_Q^{Q(\overline{\theta})} V_Q(y)\, dy \\ &= m(\overline{\theta}) + \overline{\theta} + V(Q(\overline{\theta})) - V(Q(\theta)). \end{aligned} \tag{6}$$

Since $m(\overline{\theta}) = m(\theta) = 0$, this becomes

$$\theta = \overline{\theta} + V(Q(\overline{\theta})) - V(Q(\theta)),$$

and rearranging terms and inverting V, this reduces to the expression for Q in Proposition 14. Agent types $\theta \leq \theta^*$, on the other hand, are all pooled at the point $(\overline{Q}, \theta^*)$ on the indifference curve $\overline{V}$.

One way to deal with the problem of two free endpoints $\mathcal{V}(\underline{\theta})$ and $\mathcal{V}(\overline{\theta})$ is to fix one $\mathcal{V}(\overline{\theta})$ and consider the solution to the control theory problem with only one free endpoint. Since $m(\overline{\theta}) = 0$, it follows that $\mathcal{V}(\overline{\theta}) = V(Q(\overline{\theta}))$, so $V_Q > 0$ implies that picking $\mathcal{V}(\overline{\theta})$ is equivalent to picking $Q(\overline{\theta})$. The principal's expected utility as a function of $Q(\overline{\theta})$ is

$$\begin{aligned} &[U(V^{-1}(V(Q(\overline{\theta})) + \overline{\theta} - \theta^*) - C(V^{-1}(V(Q(\overline{\theta})) + \overline{\theta} - \theta^*)) \\ &-\theta^*]F(\theta^*) + \int_{\theta^*}^{\overline{\theta}} [U(V^{-1}(V(Q(\overline{\theta})) + \overline{\theta} - x)) \\ &-C(V^{-1}(Q(\overline{\theta})) + \overline{\theta} - x]f(x)\, dx, \end{aligned}$$

where θ^* is also an implicit function of $Q(\overline{\theta})$. Then the optimal contract is the result of picking the optimal $Q(\overline{\theta})$. While the first order necessary conditions for this maximization problem are complicated, a more intuitive line of reasoning establishes that $Q(\overline{\theta})$ is bounded above by the first best level of quality $Q^*(\theta) = Q^*$, which is constant by the quasi-linearity of the principal's utility function.

Proposition 15 *If the utility functions of the principal and agent are quasi-linear then the incentive compatible mechanism* $\theta \mapsto (Q(\theta), m(\theta) + \theta)$ *is optimal only if* $Q(\overline{\theta}) \leq Q^*$.

Proof: (Appendix A) It is supposed, contrary to the proposition, that there is an optimal incentive compatible mechanism $(Q(\theta), m(\theta) + \theta)$ with endpoint $Q(\overline{\theta}) > Q^*$. Letting $(Q'(\theta), m'(\theta) + \theta)$ denote the mechanism constructed in Proposition 14 with endpoint $Q'(\overline{\theta}) = Q^*$, it is shown that the mechanism $(Q'(\theta), m'(\theta) + \theta)$ does strictly better for the principal, so that the original mechanism could not have been optimal.

An important implication of Proposition 14 is that neither cost-plus nor fixed-price contracting are ever optimal, as is shown in the following two corollaries.

Corollary 16 *If the utility functions of the principal and agent are quasi-linear then the optimal cost-plus contract* $(Q^c, \overline{\theta})$ *is suboptimal.*

Proof: The optimal cost-plus contract is the solution to the problem

$$\max_Q \int_{\underline{\theta}}^{\overline{\theta}} (U(Q) - C(Q) - \overline{\theta}) f(x)\, dx,$$

which is just $(Q^*, \overline{\theta})$. This is incentive compatible, however, and since $Q^* < \overline{Q}$, it can be seen that it does not satisfy the conditions of Proposition 14. □

Corollary 17 *If the utility functions of the principal and agent are quasi-linear then the optimal menu of fixed price contracts* $\{(Q^f(\theta, B^f), m^f(\theta, B^f) + \theta) \in \Re^2_+ | \theta \in \Theta\}$*, is suboptimal.*

Proof: (Appendix A) The proof considers two cases: constraint $m^f(\theta, B^f) \geq 0$ may or may not be slack for all agent types in $[\underline{\theta}, \overline{\theta})$. In the first case, the comparative statics of Section 3 imply that $\forall \theta \in \Theta\ Q^f_\theta = 0$ and $m^f_\theta + 1 = 0$, or in other words, the fixed price menu of contracts is the point $(Q^f(\overline{\theta}, B^f), m^f(\overline{\theta}, B^f) + \overline{\theta})$. But this is just a cost-plus contract, so suboptimality follows from Corollary 16.

In the second case, it is shown that the fixed price menu of contracts is incentive compatible and that there is an interval $[\theta', \overline{\theta}]$ with $\theta' < \overline{\theta}$ such that $\forall \theta \in [\theta', \overline{\theta}]\ \mathcal{V}_\theta = -V_Q / C_Q$. But this is not the case for the optimal incentive compatible contract, which requires that $\mathcal{V}_\theta = -1$. In other words, the utility of agent types decreases at the wrong rate for a fixed price contract.

4 Decentralization through a Menu of Linear Contracts

So far, a contract has been represented as a single point $(Q(\theta), m(\theta) + \theta) \in \Re^2_+$, but it may be possible to "decentralize" the optimal menu of contracts with a menu $\{(a(\theta), b(\theta)) \in \Re^2 | \theta \in \Theta\}$ in which each contract gives the parameters of an affine linear function

$y = a - bx$, where y is the amount of misallocated funds plus fixed costs allowable when variable cost is x. Rewriting (6), the optimal incentive compatible mechanism must satisfy

$$m(\theta) + \theta = \begin{cases} \overline{\theta} + V(Q(\overline{\theta})) - V(Q(\theta)) & \text{if } \theta \geq \theta^* \\ \theta^* & \text{if } \theta \leq \theta^*, \end{cases}$$

where $\forall \theta \geq \theta^*$ $Q_\theta < 0$ and $\forall \theta \leq \theta^*$ $Q_\theta = 0$. It follows that the optimal amount of misallocated funds plus fixed cost can be considered a function of the quality level Q, written

$$\hat{M}(Q) = \begin{cases} \overline{\theta} + V(Q(\overline{\theta})) - V(Q) & \text{if } \theta \geq \theta^* \\ \theta^* & \text{if } \theta \leq \theta^*. \end{cases}$$

Then, using the assumption that $C_Q > 0$, define $M(C) \equiv \hat{M}(C^{-1}(C))$ and note that $\forall \theta > \theta^*$

$$\begin{aligned} M_C &= \hat{M}_Q C_C^{-1} = -\frac{V_Q}{C_Q} < 0 \\ M_{CC} &= \hat{M}_{QQ} C_C^{-1} + \hat{M}_Q C_{CC}^{-1} \\ &= -\frac{V_{QQ}}{C_Q} + \frac{V_Q C_{QQ} (C_C^{-1})^2}{C_Q} > 0, \end{aligned}$$

where $C_C^{-1} = 1/C_Q$ and $C_{CC}^{-1} = -(C_{QQ})(C_C^{-1})^2$. Of course, $\forall \theta < \theta^*$ we have $M_C = 0$ and $M_{CC} = 0$. Since M is convex, the optimal menu of contracts $\{(Q(\theta), m(\theta) + \theta) \in \Re_+^2 | \theta \in \Theta\}$ can be implemented by the menu of contracts $\{(a(\theta), b(\theta)) \in \Re^2 | \theta \in \Theta\}$, in which each contract is an affine linear function $M = a(\theta) - b(\theta)C$, where $\forall \theta \geq \theta^*$

$$\begin{aligned} a(\theta) &= \overline{\theta} + V(Q(\overline{\theta})) - V(Q(\theta)) + \frac{V_Q(Q(\theta))}{C_Q(Q(\theta))} C(Q(\theta)) \\ b(\theta) &= V_Q(Q(\theta)) C_C^{-1}(C(Q(\theta))) = \frac{V_Q(Q(\theta))}{C_Q(Q(\theta))} \end{aligned}$$

and $\forall \theta \leq \theta^*$ $a(\theta) = a(\theta^*)$ and $b(\theta) = b(\theta^*)$.

To see that the menu of linear contracts $\{(a(\theta), b(\theta)) \in \Re^2 | \theta \in \Theta\}$ is incentive compatible and induces truthful agents to pick the optimal levels of misallocated funds and quality, consider the maximization problem confronting a type θ agent:

$$\max_{\hat{\theta}, Q} V(Q) + a(\hat{\theta}) - b(\hat{\theta}) C(Q) - \theta,$$

or after substituting for a and b,

$$\max_{\hat{\theta}, Q} V(Q) + \overline{\theta} + V(Q(\overline{\theta})) - V(Q(\hat{\theta})) + \frac{V_Q(Q(\hat{\theta}))}{C_Q(Q(\hat{\theta}))} C(Q(\hat{\theta})) - \frac{V_Q(Q(\hat{\theta}))}{C_Q(Q(\hat{\theta}))} C(Q) - \theta.$$

The first order condition for Q is

$$V_Q(Q) - \frac{V_Q(Q(\hat{\theta}))}{C_Q(Q(\hat{\theta}))} C_Q(Q) = 0,$$

or after manipulation,

$$\frac{V_Q(Q)}{C_Q(Q)} = \frac{V_Q(Q(\hat{\theta}))}{C_Q(Q(\hat{\theta}))},$$

and since V_Q/C_Q is a strictly decreasing function this implies that $Q = Q(\hat{\theta})$. That is, the agent picks the optimal level of quality corresponding to his reported type.

After substituting $Q(\hat{\theta})$ for Q, the first order condition for $\hat{\theta}$ becomes

$$\frac{V_Q(Q(\hat{\theta}))}{C_Q(Q(\hat{\theta}))} = \frac{V_Q(Q(\hat{\theta}))}{C_Q(Q(\hat{\theta}))},$$

which is satisfied by all values of $\hat{\theta}$, and in particular it is satisfied by $\hat{\theta} = \theta$, so the decentralized mechanism is incentive compatible. When an agent of type θ reports the truth he picks quality level $Q(\theta)$, and to see that he also picks the optimal level of misallocated funds plus fixed cost equal, consider the linear contract of the type $\theta \geq \theta^*$ agent:

$$\begin{aligned}
&a(\theta) - b(\theta)C(Q(\theta)) \\
&= \overline{\theta} + V(Q(\overline{\theta})) - V(Q(\theta)) + \frac{V_Q(Q(\theta))}{C_Q(Q(\theta))}C(Q(\theta)) - \frac{V_Q(Q(\theta))}{C_Q(Q(\theta))}C(Q(\theta)) \\
&= \overline{\theta} - V(Q(\overline{\theta})) - V(Q(\theta)) \\
&= \theta,
\end{aligned}$$

where the last step follows after substituting for $Q(\theta)$ from Proposition 14. A similar argument establishes that a type $\theta \leq \theta^*$ agent picks $m(\theta) + \theta = \theta^*$. Therefore, the decentralized mechanism does indeed induce truthful agents to pick the optimal levels of misallocated funds and quality.

Here, a cost-plus contract is just the case $a = \overline{\theta}$ and $b = 0$, and a fixed price contract is just the case $a = B^f$ and $b = 1$. Since $b = V_Q C_C^{-1} > 0$, it can be seen that no agent is faced with the cost-plus contract, but it is not clear whether any agent faces a fixed price contract. Note that $\forall \theta \geq \theta^*$

$$\frac{db}{d\theta} = \frac{C_Q V_{QQ} Q_\theta - V_Q C_{QQ} Q_\theta}{(C_Q)^2} > 0,$$

so θ can be considered an implicit function of b with $\frac{d\theta}{db} = 1/\frac{db}{d\theta} > 0$. Then $\frac{da}{db} = C$ and $\frac{d^2 a}{db^2} = C_Q Q_\theta / \frac{db}{d\theta} < 0$. That is, a higher coefficient of cost sharing b must be compensated by a higher fixed payment a, and the fixed payment is a concave function of the slope of the linear contract.

5 The Adversarial Model

A standard model of adverse selection very similar to that presented in Section 1 is the adversarial model, offered by Laffont and Tirole (Chapter 2, [5]) in an analysis of the procurement of a public project from a private firm by a regulator. Let $Q \in \Re_+$ denote the quality of the project; let $t \in \Re$ denote a monetary transfer from the principal (the regulator) to the agent (a firm); and let the cost $\mathcal{C}$ of the project be given by $Q(m+\theta)$, where $\theta \in [\underline{\theta}, \overline{\theta}] \subset \Re_+$ is a random variable observed by the agent but not by the principal, whose beliefs are given by the non-atomic distribution function F with density f, and $m \in \Re$ is a moral hazard variable chosen by the agent but unobserved by the principal. Then the utility of the principal is given by $U(Q) - Q(m+\theta) - t$ and the utility of the agent is $\psi(m) + t$, where it is assumed that U and ψ are twice continuously differentiable with $U_Q > 0$, $U_{QQ} \leq 0$, $\psi_m > 0$, $\psi_{mm} \leq 0$, and $\lim_{m \to -\theta} \psi(m) = -\infty$. In this model, m is interpreted as slack—or "negative effort"—so that the principal prefers lower values, and a high θ corresponds to higher costs.[4]

As in the cooperative model, the principal observes quality and cost, so once the information revelation problem is solved he can solve for $m = (\mathcal{C}/Q) - \theta$. Letting V denote the utility of the agent, the utility function of the principal can be written $U(Q) - \theta - t - Q\psi^{-1}(V - t)$, so it is apparent that—unlike the cooperative model—the principal's interests are in conflict with those of the agent. That is, the principal does best when V is infinitely negative. Because of the voluntary nature of market participation, however, the principal must offer the agent at least as much utility as the agent's next best alternative. This reservation utility level is assumed to be the same for all types of agent and is normalized to zero.

Now consider a more general model, in which cost is a function $\mathcal{C}(Q, m+\theta)$ of project quality and the sum of the slack variable and random shock, the principal's utility is given by $U(Q) - \mathcal{C}(Q, m+\theta) - t$, and the agent's utility is given by $V(Q, m) + t$, where $\mathcal{C}$, U, and V are twice continuously differentiable with $\mathcal{C}_Q > 0$, $\mathcal{C}_m > 0$, $U_Q > 0$, $U_{QQ} \leq 0$, $V_m > 0$, $V_{mm} \leq 0$, $V_Q \geq 0$, $V_{QQ} \leq 0$, and $V_{Qm} \geq 0$. Furthermore, it is assumed that the functions V and $\mathcal{C}$ and variables m and t satisfy a set of constraints $\mathrm{CON}(V, \mathcal{C}, m, t)$. It can then be seen that the cooperative model is a special case of the general model when

[4] In Laffont and Tirole's original formulation the principal's objective function includes the rent of the agent, and the shadow cost of transfering one dollar to the agent is $(1+\lambda)$ dollars, where $\lambda > 0$ is interpreted as a social cost due to distortionary taxation. Neither of these features serves the purposes of this section, so they have been dropped. The moral hazard variable is originally effort $e = -m$, but this redefinition is inconsequential and serves to unify notation.

CON = COOP, where

$$\text{COOP}(V,\mathcal{C},m,t) = \begin{cases} V(Q,m) = V(Q) + m \\ \mathcal{C}(Q,m+\theta) = C(Q) + m + \theta \\ C_Q > 0 \\ C_{QQ} > 0 \\ m \geq 0 \\ t = 0, \end{cases}$$

and the adversarial model is a special case when CON = ADV, where

$$\text{ADV}(V,\mathcal{C},m,t) = \begin{cases} V(Q,m) = \psi(m) \\ \psi_m > 0 \\ \psi_{mm} \leq 0 \\ \lim_{m\to-\theta} \psi(m) = -\infty \\ \mathcal{C}(Q,m+\theta) = Q(m+\theta) \\ V(Q,m) + t \geq 0. \end{cases}$$

The principal's problem under COOP is solved as in Section 3, with the additional constraint $t = 0$ and corresponding Lagrange multiplier λ. The partials of the Hamiltonian with respect to Q and m are then

$$\begin{aligned} U_Q &= \mathcal{C}_Q - V_Q + \lambda V_Q/f \\ \lambda &= \gamma, \end{aligned}$$

respectively, which reduce to condition (5).

The principal's optimal contracting problem under ADV can be analyzed as a control theory problem with state variable $\mathcal{V}(\theta) = \psi(m(\theta)) + t(\theta)$ and control variables Q, m, and t, in which the transfer can be written as an implicit function $t(\mathcal{V}, Q, m) = \mathcal{V} - \psi(m)$ and the first order incentive compatibility constraint reduces to $\mathcal{V}_\theta = -\psi_m$. Guesnerie and Laffont (Corollary 2.1, [3]) and Laffont and Tirole (Proposition 1.2, [5]) show that this first order constraint in conjunction with the second order constraint $m_\theta \geq -1$ is necessary and sufficient for incentive compatibility, so the relaxed problem can be solved

$$\max_{Q,m} \int_{\underline{\theta}}^{\overline{\theta}} [U(Q) - \mathcal{C}(Q, m + x)] f(x)\, dx$$

subject to $\text{ADV}(V,\mathcal{C},m,t)$ and $\mathcal{V}_\theta = -V_m$, leaving to be determined conditions under which the relaxed solution also satisfies $m_\theta \geq -1$. Letting μ be the multiplier for the first order constraint, $\mu_\theta = f$ and the transversality condition $\mu(\underline{\theta}) = 0$ imply that $\mu = F$. From the maximum principle, we have

$$\begin{aligned} U_Q &= \mathcal{C}_Q && (7) \\ V_m &= \mathcal{C}_m + FV_{mm}/f, && (8) \end{aligned}$$

or after substitution from ADV,

$$
\begin{aligned}
U_Q &= m + \theta \\
\psi_m &= Q + F\psi_{mm}/f.
\end{aligned}
$$

It can then be seen that the second order condition for incentive compatibility is in fact met by the solution to the relaxed problem when the following restrictions are met:

$$\psi_{mmm} \geq 0,\ \psi_{mm}U_{QQ} < 1,\ \frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right) \geq 0.$$

Under these conditions, the optimal contract is determined by (7) and (8), implying that the principal can achieve the first best provision of quality but must accept the informational wedge $F\psi_{mm}/f \leq 0$ between the marginal benefit ψ_m of slack and the marginal cost Q.

It is of interest to consider the ramifications of coincidence of interest for the principal's contracting problem in the adversarial model. Abstracting from the details of the adversarial model, this can be done by replacing ADV with ADV*, where

$$\text{ADV}^*(V, \mathcal{C}, m, t) = \begin{cases} \lim_{m \to -\theta} V(Q, m) = -\infty \\ V(Q, m) + t \geq 0. \end{cases}$$

Again, the principal's optimal contracting problem is a control theory problem with state variable $\mathcal{V}(\theta) = V(Q(\theta), m(\theta)) + t(\theta)$ and control variables Q, m, and t, where $t(\mathcal{V}, Q, m) = \mathcal{V} - V(Q, m)$ and the first order incentive compatibility constraint reduces to $\mathcal{V}_\theta = -V_m$. The Hamiltonian is

$$H = [U(Q) - \mathcal{C}(Q, m + \theta) - \mathcal{V} + V(Q, m)]f - \mu V_m(Q, m)$$

with partials yielding

$$
\begin{aligned}
U_Q + V_Q &= \mathcal{C}_Q + FV_{Qm}/f && (9) \\
V_m &= \mathcal{C}_m + FV_{mm}/f. && (10)
\end{aligned}
$$

For the purposes of exposition, it is assumed that the second order incentive compatibility constraints $Q_\theta \leq 0$ and $t_\theta \leq 0$ are not binding at the optimum, so that the solution to the system of equations (9) and (10) is globally incentive compatible.

Comparing (8) with (10), it can be seen that the marginal condition which determines the level of slack under ADV is unaffected by the addition of coincidence of interest. After comparing (7) with (9), this is obviously not true of the optimal level of quality. With coincidence of interest, an increase in the level of quality affects the marginal utility of the principal in three ways: it increases the cost of the project by $\mathcal{C}_Q$; it raises the utility of the agent by V_Q, and this raises the principal's utility by V_Q through a correspondingly lower transfer; and since $V_{Qm} \geq 0$, slacking becomes relatively more attractive to the agent. Whether the principal is better off depends on the size of V_Q relative to FV_{Qm}/f and how the level of slacking is affected through $\mathcal{C}_{Qm}$ and V_{Qmm}. In the special case of additive separability ($V_{Qm} = V_{Qmm} = \mathcal{C}_{Qm} = 0$), however, the principal is strictly better off with coincidence of interest: in effect, the agent is willing to pay marginal costs up to V_Q for extra quality.

The explicit monetary transfer t plays an important role in optimal contracting subject to the constraints ADV*, and for this reason the above results do not apply to contracting with a non-profit agent. To see how non-profit agency affects the optimal contract, define the constraint set ADV** by adding the constraint $t = 0$ to ADV* and omiting the individual rationality constraint $V(Q, m) + t \geq 0$. The control theory problem is unchanged save for the constraint $t = 0$ with Lagrange multiplier λ and the presence of two free endpoints rather than one, so the Hamiltonian is now

$$H = [U(Q) - \mathcal{C}(Q, m + \theta) - \mathcal{V} + V(Q, m)]f - \mu V_m + \lambda[\mathcal{V} - V(Q, m)]$$

with partials yielding

$$\begin{aligned} U_Q + V_Q &= \mathcal{C}_Q + [\mu V_{Qm} + \lambda V_Q]/f & (11) \\ V_m &= \mathcal{C}_m + [\mu V_{mm} + \lambda V_m]/f, & (12) \end{aligned}$$

where $\mu(\theta) = F(\theta) - \int_{\underline{\theta}}^{\overline{\theta}} \lambda(x)\, dx$. Of course, the principal's expected utility can be no higher when t is fixed at zero, but it does not follow from (11) and (12) that for any given value of θ the principal is worse off. Although in expectation the principal's utility is lower, he may be better off for some values of θ than he would have been otherwise.

The optimal contract under constraints ADV** is difficult to intuit from (11) and (12) without the assumption of quasi-linear preferences of the agent, for while control theoretic analysis offers an interpretation of the multiplier μ of the incentive compatibility constraint, it has little to say about the Lagrange multiplier λ. The assumption of quasi-linearity, however, is inconsistent with the constraints ADV**. The analytical advantage of the cooperative model is a vast simplification of the form of the optimal contract, given by (11) and (12) after dropping the assumption that $\lim_{m \to -\theta} V(Q, m) = -\infty$ and adding the assumptions of quasi-linear cost, quasi-linear preferences, and non-negativity of m.

6 Conclusion

After presenting the details of the model with coincidence of interest, cost-plus and fixed price contracting have been defined and compared, and the form of the optimal contract has been determined for the case of quasi-linear preferences. It was found that the optimal menu of contracts for 'high' types of agent coincides with the agent's indifference curve through $(Q(\overline{\theta}), \overline{\theta})$, while 'low' types of agent are pooled at $(\overline{Q}, \theta^*)$. Subsequently, it was shown that cost-plus and fixed price contracting are suboptimal, and the possibility of decentralizing the optimal mechanism through a menu of linear contracts was confirmed. Lastly, the assumptions underlying the cooperative model were compared with those of the adversarial model adapted from Laffont and Tirole.

The approach here is of analytical interest because it applies to a class of contracting problems that the standard models of principal agent theory have left largely untouched, allowing a non-profit agent to have a personal stake in his work. The reformulation of the moral hazard variable used by this approach leads to the difficulty that the necessary first order conditions for an interior maximum need not hold for any type of agent, complicating the task of finding a form of the incentive compatibility constraint which is amenable to control theoretic analysis. This reformulation is not without its rewards, however, for it also leads to an extremely simple form of the optimal contract for the case of quasi-linear preferences.

A Proofs of Propositions

Proposition 11 *When the agent has quasi-linear utility $V(Q,m) = V(Q) + m$, the following conditions are sufficient for incentive compatibility:* $\forall \theta \in \Theta$

I $Q_\theta \leq 0$

II $m_\theta + 1 \geq 0$

III $V_Q Q_\theta + m_\theta + 1 \leq 0$

IV $m(\theta)(V_Q Q_\theta + m_\theta + 1) = 0.$

Proof: Suppose in order to show a contradiction that incentive compatibility does not hold, so that $\exists \theta', \theta'' \in \Theta$ such that θ' is better off imitating θ'' and, moreover, he can do so successfully. That is,

$$V(Q(\theta'')) + m(\theta'') + \theta'' - \theta' > V(Q(\theta')) + m(\theta')$$

and

$$\theta' \leq m(\theta'') + \theta''.$$

There are two cases to consider: $\theta' < \theta''$ and $\theta' > \theta''$.

Case 1: Because the agents have quasi-linear utility, they share the same indifference maps over Q and $m+\theta$, up to a translation of the origin. Let V' denote the indifference curve through $(Q(\theta'), m(\theta') + \theta')$ and let V'' denote the indifference curve through $(Q(\theta''), m(\theta'') + \theta'')$, as pictured in Figure 6 (top).

[Figure 6 about here.]

Since θ' prefers $(Q(\theta''), m(\theta'') + \theta'')$ to $(Q(\theta'), m(\theta') + \theta')$, we know that V'' lies above and to the right of V'. Furthermore, we know that $Q(\theta') \geq Q(\theta'')$ since $Q_\theta \leq 0$, and that $m(\theta') + \theta' \leq m(\theta'') + \theta''$ since $m_\theta + 1 \geq 0$.

The assumptions on V guarantee that V' is downward sloping and smooth, so it can be taken as the graph of a strictly decreasing, continuously differentiable function $q : D' \to \Re_+$, where $D' = \{x \in M | V(\mathcal{Q}(x)) + x - \theta' = V(Q(\theta')) + m(\theta')\}$ with derivative $q_x(m(\theta) + \theta) = -1/V_Q(Q(\theta)) < 0$. Note from our assumptions on V that $x^* = \max\{x \in D'\}$ is well-defined and $q(x^*) = 0$. Note also that, using the quasi-linearity of V, we can take V'' to be the image of $D'' = \{x \in M | V(\mathcal{Q}(x)) + x - \theta'' = V(Q(\theta'')) + m(\theta'')\}$ under the function $q(x - \alpha)$, where α is some positive constant. It follows that $\mathcal{Q}(x') = q(x')$ and $\mathcal{Q}(x'') = q(x'' - \alpha)$, where $x' = m(\theta') + \theta'$ and $x'' = m(\theta'') + \theta''$.

The first step is to show that $\forall x \in [x', x^*]$ $q(x) \geq \mathcal{Q}(x)$, so suppose not in order to show a contradiction. Then $\exists \hat{x} \in [x', x^*]$ with $\hat{x} > x'$ such that $\mathcal{Q}(\hat{x}) > q(\hat{x})$. This implies that $\mathcal{Q}(\hat{x}) - \mathcal{Q}(x') > q(\hat{x}) - q(x')$. Since $\mathcal{Q}$ is non-increasing on the finite interval $[x', x^*]$, it follows that $\mathcal{Q}_x$ exists almost everywhere on the interval and that

$$\begin{aligned} \int_{x'}^{\hat{x}} \mathcal{Q}_x(x)\, dx &\geq \mathcal{Q}(\hat{x}) - \mathcal{Q}(x') \\ &> q(\hat{x}) - q(x') = \int_{x'}^{\hat{x}} q_x(x)\, dx. \end{aligned}$$

Now, let $\tilde{M} = \{x \in M | \mathcal{Q}_x(x) > q_x(x)\}$, and to see that $\tilde{M}$ has positive measure, suppose not. Then from the previous inequality we have

$$\int_{[x', \hat{x}] \setminus \tilde{M}} \mathcal{Q}_x(x)\, dx > \int_{[x', \hat{x}] \setminus \tilde{M}} q_x(x)\, dx,$$

but this contradicts the fact that $\forall x \in [x', \hat{x}] \setminus \tilde{M}$ $q_x(x) \geq \mathcal{Q}_x(x)$. Therefore, $\tilde{M}$ has positive measure. Moreover, $\exists x \in \tilde{M} \setminus P_\Lambda$ since Footnote 3 requires that P_Λ is countable, and therefore has measure zero.

Pick any non-pooling point $\tilde{x} \in \tilde{M} \setminus P_\Lambda$, so that $m(\theta) + \theta = \tilde{x}$ implies $m_\theta(\theta) + 1 > 0$. Then there is a unique $\tilde{\theta} \in \Theta$ such that $m(\tilde{\theta}) + \tilde{\theta} = \tilde{x}$ and

$$\mathcal{Q}_x(\tilde{x}) = \frac{Q_\theta(\tilde{\theta})}{m_\theta(\tilde{\theta}) + 1}.$$

Since $\tilde{x} \in \tilde{M}$, we know

$$\frac{Q_\theta(\tilde{\theta})}{m_\theta(\tilde{\theta}) + 1} = \mathcal{Q}_x(\tilde{x}) > q_x(\tilde{x}) = -\frac{1}{V_Q(Q(\tilde{\theta}))},$$

or equivalently,

$$V_Q(Q(\tilde{\theta}))Q_\theta(\tilde{\theta}) + m_\theta(\tilde{\theta}) + 1 > 0,$$

which contradicts the assumptions of the proposition. Therefore, we have $\forall x \in [x', x^*]\ q(x) \geq \mathcal{Q}(x)$.

The next step is to show that $x'' \notin D'$. If, contrary to the claim, $x'' \in D'$ then by construction $x'' \in [x', x^*]$, so from $q_x < 0$ and $a > 0$ it follows that $\mathcal{Q}(x'') = q(x'' - a) > q(x'') \geq \mathcal{Q}(x'')$, a contradiction.

The final step is to suppose that $x'' \notin D'$, as in Figure 6 (bottom). Then by construction $x'' > x^*$. Note that $\mathcal{Q}(x^*) = 0$, since $0 \leq \mathcal{Q}(x^*) \leq q(x^*) = 0$, so from $\mathcal{Q} \geq 0$ and $\mathcal{Q}_x \leq 0$ it follows that $\forall x \in [x^*, x'']\ \mathcal{Q}_x(x) = 0$. That is, the graph of $\mathcal{Q}$ must be flat along the x-axis between x^* and x''. To see that this is impossible, define the interval $\Theta' = \{\theta \in \Theta | x^* < m(\theta) + \theta \leq x''\}$ and note that we have $\forall \theta \in \Theta'\ Q_\theta(\theta) = 0$. Since it is assumed that $V_Q Q_\theta + m_\theta + 1 \leq 0$ and $m_\theta + 1 \geq 0$, we then have $\forall \theta \in \Theta'\ m_\theta(\theta) + 1 = 0$. But this gives us

$$x'' - x^* = \int_{x^*}^{x''} dx = \int_{\Theta'} m_\theta(x) + 1\, dx = 0,$$

or in other words, $x'' = x^* \in D'$, a contradiction.

Case 2: Now suppose that $\exists \theta', \theta'' \in \Theta$ such that $\theta'' < \theta'$ and the type θ' agent is better off imitating θ''. Moreover, he can do so successfully, so that $\theta' \leq m(\theta'') + \theta''$. Note that $\forall \theta \in [\theta'', \theta')\ m(\theta) > 0$. To see this, we know that for such a θ, $m(\theta) + \theta \geq m(\theta'') + \theta''$ since $m_\theta + 1 \geq 0$. We then have $\theta < \theta' \leq m(\theta'') + \theta'' \leq m(\theta) + \theta$. Subtracting θ from the first, second, and fourth terms yields

$$0 < \theta' - \theta \leq m(\theta).$$

Then by the assumptions of the proposition we have $\forall \theta \in [\theta'', \theta')$

$$V_Q(\theta) Q_\theta(\theta) + m_\theta(\theta) + 1 = 0.$$

That is, the necessary first order conditions for an interior maximum hold for all such types of agent.

In the case of general preferences, we can write the supposition that the agent of type θ' prefers the type θ'' contract to his own as

$$\begin{aligned}
&V(Q(\theta''), m(\theta'') + \theta'' - \theta') - V(Q(\theta'), m(\theta')) \\
&= -\int_{\theta''}^{\theta'} V_Q(Q(x), m(x) + x - \theta') Q_\theta(x) \\
&\quad + V_m(Q(x), m(x) + x - \theta')(m_\theta(x) + 1)\, dx \\
&= -\int_{\theta''}^{\theta'} V_m \left(\frac{V_Q}{V_m} Q_\theta + m_\theta + 1 \right) dx > 0.
\end{aligned}$$

Define

$$V(\hat{\theta}, \theta) = V(Q(\hat{\theta}), m(\hat{\theta}) + \hat{\theta} - \theta),$$

where the first component of $V(\hat{\theta}, \theta)$ represents the agent's reported type and the second component represents the agent's true type. The supposition can be rewritten as

$$\int_{\theta''}^{\theta'} V_m(x, \theta') \left(\frac{V_Q(x, \theta')}{V_m(x, \theta')} Q_\theta(x) + m_\theta(x) + 1 \right) dx < 0,$$

and the first order conditions become $\forall\theta \in [\theta'', \theta')$

$$\frac{V_Q(\theta,\theta)}{V_m(\theta,\theta)} Q_\theta(\theta) + m_\theta(\theta) + 1 = 0.$$

Next, note that the assumptions on preferences of the agent imply that his utility function possesses the single crossing property:

$$\frac{\partial}{\partial\theta}\left(\frac{V_Q(\hat\theta,\theta)}{V_m(\hat\theta,\theta)}\right) = \frac{-V_m V_{Qm} + V_Q V_{mm}}{V_m^2} \le 0.$$

The single crossing property together with the first order condition evaluated at θ implies $\forall\theta \in [\theta'', \theta')$

$$V_m(\theta,\theta')\left(\frac{V_Q(\theta,\theta')}{V_m(\theta,\theta')} Q_\theta(\theta) + m_\theta(\theta) + 1\right) \ge 0.$$

Then we have

$$\int_{\theta''}^{\theta'} V_m(x,\theta')\left(\frac{V_Q(x,\theta')}{V_m(x,\theta')} Q_\theta(x) + m_\theta(x) + 1\right) dx \ge 0,$$

contradicting the above formulation of our supposition. □

Proposition 12 *If the utility functions of the principal and agent are quasi-linear*

$$\begin{aligned} U(Q,B) &= U(Q) - B \\ V(Q,m) &= V(Q) + m \end{aligned}$$

and the mapping $\theta \mapsto (Q(\theta), m(\theta) + \theta)$ is an optimal incentive compatible mechanism then the following conditions hold: $\forall\theta \in \Theta$

I $Q_\theta \le 0$

II $V_Q Q_\theta + m_\theta + 1 = 0.$

Proof: For both parts of the proposition, the method of proof will be to perturb the mechanism $\theta \mapsto (Q(\theta), m(\theta) + \theta)$ in order to find a new mechanism $(Q'(\theta), m'(\theta) + \theta)$ which does strictly better for the principal. First, consider (I).

[Figure 7 about here.]

Suppose in order to show a contradiction that $\exists\theta' \in \Theta$ such that $Q_\theta(\theta') > 0$, as depicted in Figure 7 (top). Since Q_θ is continuous, $\exists\delta' > 0$ such that $\forall\theta \in B_{\delta'}(\theta')$ $Q_\theta(\theta) > 0$. Let the constant $\overline{Q}$ be uniquely defined by $U(\overline{Q}) + V(\overline{Q}) - C(\overline{Q}) \equiv 0$. Next, we need to show that there is an open set $B' \subset B_{\delta'}(\theta')$ the image of which under Q lies either entirely above Q^* or entirely below it. There are two cases to consider: $Q(\theta') \le \overline{Q}$ and $Q(\theta') \ge \overline{Q}$. Since the proofs of the two cases are symmetric, we will assume without loss of generality that $Q(\theta') \le \overline{Q}$.

Note that there is a type θ'' arbitrarily close to θ' such that $Q(\theta'') \neq \overline{Q}$. To see this, suppose that $\exists \epsilon > 0$ such that $\forall \theta \in B_\epsilon(\theta')$ $Q(\theta) = \overline{Q}$. But then $\forall \theta \in B_\epsilon(\theta)$ $Q_\theta = 0$, and in particular, $Q_\theta(\theta') = 0$, a contradiction. Then we can take $\theta'' \in B_{\delta'}(\theta')$ such that $Q(\theta'') \neq \overline{Q}$, and again without loss of generality let $Q(\theta'') < \overline{Q}$, so by the continuity of Q we know $\exists \delta'' > 0$ such that $\forall \theta \in B_{\delta''}(\theta'')$ $Q(\theta) < \overline{Q}$. We then have our open set $B = B_{\delta'}(\theta') \cap B_{\delta''}(\theta'')$. That is, $\forall \theta \in B$ $Q_\theta(\theta) > 0$ and $Q(\theta) < \overline{Q}$.

Now we will show that $\forall \theta \in B$ $m(\theta) > 0$. Let $B = (\theta_1', \theta_2')$, take $\theta \in B$, and note that from incentive compatibility and Proposition 10 we have

$$V(Q(\theta)) + m(\theta) \geq V(Q(\theta_2')) + m(\theta_2') + \theta_2' - \theta,$$

which implies

$$m(\theta) + \theta - m(\theta_2') - \theta_2' \geq V(Q(\theta_2')) - V(Q(\theta)) > 0,$$

where the last inequality follows from $Q_\theta > 0$ and $V_Q > 0$. After rearranging terms, we have

$$m(\theta) > m(\theta_2') + \theta_2' - \theta > m(\theta_2') \geq 0,$$

where the second inequality follows from $\theta < \theta_2'$.

Now define $\overline{B} = [\theta_1, \theta_2] \subset B$ with $\theta_1 < \theta_2$ and consider the new contract

$$(Q'(\theta), m'(\theta) + \theta) = \begin{cases} (Q(\theta_2), m(\theta_2) + \theta_2) & \text{if } \theta \in \overline{B} \\ (Q(\theta), m(\theta) + \theta) & \text{else,} \end{cases}$$

as in Figure 7 (bottom). To see that $(Q', m' + \theta)$ is incentive compatible, suppose it is not, leaving only two cases to be considered: a type $\theta' \notin \overline{B}$ agent wants to imitate a type $\theta'' \in \overline{B}$ agent and can do so successfully, and a type $\theta'' \in \overline{B}$ agent wants to imitate a type θ' agent and can do so successfully. The first case can be ruled out immediately, since the type $\theta' \notin \overline{B}$ agent would have imitated the type θ_2 agent under the original mechanism, violating incentive compatibility of $(Q, m + \theta)$.

In the second case, by Proposition 10 we have

$$V(Q(\theta_2)) + m(\theta_2) + \theta_2 - \theta'' < V(Q(\theta')) + m(\theta') + \theta' - \theta'',$$

but by the incentive compatibility of $(Q, m + \theta)$ we also have

$$V(Q(\theta'')) + m(\theta'') \geq V(Q(\theta')) + m(\theta') + \theta' - \theta''.$$

Conjoining the first inequality with the second, we have

$$\begin{aligned} & V(Q(\theta'')) + m(\theta'') + \theta'' - V(Q(\theta_2)) - m(\theta_2) - \theta_2 \\ = & \int_{\theta_2}^{\theta''} V(Q(x))Q_\theta(x) + m_\theta(x) + 1\, dx > 0. \end{aligned}$$

To get our contradiction, note that $[\theta'', \theta_2] \subset B$ implies that $\forall \theta \in [\theta'', \theta_2]$ $m(\theta) > 0$, which implies that the agent's constraint is not binding for any such type, so using the complementary slackness conditions for the agent's problem, we have $\forall\theta \in [\theta'', \theta_2]$ $\mu = 0$. Then from the Kuhn-Tucker conditions for the agent's problem we have $\forall\theta \in [\theta'', \theta_2]$

$$V_Q Q_\theta + m_\theta + 1 - \mu(m_\theta + 1) = V_Q Q_\theta + m_\theta + 1 = 0.$$

But then

$$\int_{\theta''}^{\theta_2} V(Q(x))Q_\theta(x) + m_\theta(x) + 1\, dx = 0,$$

a contradiction. Therefore, the new mechanism $(Q', m' + \theta)$ is indeed incentive compatible.

It remains to be shown that the principal is strictly better off with the mechanism $(Q', m'+\theta)$. Since the new mechanism is incentive compatible the principal's payoff is

$$\begin{aligned}
&\int_{\underline{\theta}}^{\theta_1} [U(Q(x)) - C(Q(x)) - m(x) - x]f(x)\, dx \\
&+ \int_{\theta_1}^{\theta_2} [U(Q(\theta_2)) - C(Q(\theta_2)) - m(\theta_2) - \theta_2]f(x)\, dx \\
&+ \int_{\theta_2}^{\overline{\theta}} [U(Q(x)) - C(Q(x)) - m(x) - x]f(x) dx
\end{aligned}$$

and it suffices to show that

$$\begin{aligned}
&\int_{\theta_1}^{\theta_2} [U(Q(\theta_2)) - C(Q(\theta_2)) - m(\theta_2) - \theta_2]f(x)\, dx \\
&> \int_{\theta_1}^{\theta_2} [U(Q(x)) - C(Q(x)) - m(x) - x]f(x)\, dx.
\end{aligned}$$

The result will follow if we can show that $\forall x \in [\theta_1, \theta_2]$

$$U(Q(\theta_2)) - C(Q(\theta_2)) - m(\theta_2) - \theta_2 - U(Q(x)) + C(Q(x)) + m(x) + x > 0,$$

which can be rewritten as

$$\int_x^{\theta_2} U_Q(Q(y))Q_\theta(y) - C_Q(Q(y))Q_\theta(y) - m_\theta(y) - 1\, dy > 0.$$

Again, the result will follow if the integrand is positive over $[x, \theta_2]$. By the definition of $\overline{Q}$ and the assumptions on U, V, and C, we know that $\forall Q < \overline{Q}$ $U_Q(Q) + V_Q(Q) - C_Q(Q) > 0$. Picking an arbitrary $y \in [x, \theta_2] \subset B$, we have $Q(y) < \overline{Q}$ by the construction of B. Also by the construction of B, we have $Q_\theta(y) > 0$, so

$$(U_Q(Q(y)) + V_Q(Q(y)) - C_Q(Q(y)))Q_\theta(y) > 0.$$

Adding and subtracting $m_\theta(y) + 1$ to the left hand side of the inequality yields

$$\begin{aligned}
&U_Q(Q(y))Q_\theta(y) - C_Q(Q(y))Q_\theta(y) - m_\theta(y) - 1 \\
&\quad + V_Q(Q(y))Q_\theta(y) + m_\theta(y) + 1 \\
&= U_Q(Q(y))Q_\theta(y) - C_Q(Q(y))Q_\theta(y) - m_\theta(y) - 1 > 0,
\end{aligned}$$

where we use the result derived above that $\forall\theta \in B$ $V_Q Q_\theta + m_\theta + 1 = 0$. Therefore, the principal is strictly better off with the new mechanism $(Q', m' + \theta)$, so an optimal incentive compatible mechanism must satisfy $\forall\theta \in \Theta$ $Q_\theta \leq 0$.

This would conclude the first part of the proof, except that the new mechanism $(Q', m' + \theta)$ is not piecewise continuously differentiable, and since the original mechanism was only supposed to be optimal among the class of piecewise continuously differentiable mechanisms, it must be compared only to mechanisms within that class. The method of proof can be extended to find a piecewise continuously differentiable mechanism by taking the function f^ϵ, which maps $[\theta_2 - \epsilon, \theta_2]$ onto $[\theta_1, \theta_2]$, defined by

$$f^\epsilon(x) = [(x - \theta_2 + \epsilon)(\theta_2 - \theta_1)/\epsilon] + \theta_1,$$

where f is continuous in both x and ϵ. Then for $\epsilon > 0$, define the new mechanism

$$(Q^\epsilon(\theta), m^\epsilon(\theta) + \theta) = \begin{cases} (Q(\theta_2), m(\theta_2) + \theta_2) & \text{if } \theta \in [\theta_1, \theta_2 - \epsilon] \\ (Q(f^\epsilon(\theta)), m(f^\epsilon(\theta)) + \theta) & \text{if } \theta \in [\theta_2 - \epsilon, \theta_2] \\ (Q(\theta), m(\theta) + \theta) & \text{else,} \end{cases}$$

and note that $(Q^\epsilon, m^\epsilon + \theta)$ is piecewise continuously differentiable for each ϵ.

To see that $(Q^\epsilon, m^\epsilon + \theta)$ is incentive compatible, suppose an agent of type $\theta'' \in [\theta_2 - \epsilon, \theta_2]$ wants to imitate a type θ' agent and can do so successfully. Then as above, we have

$$\begin{aligned} V(Q(\theta'')) + m(\theta'') &\geq V(Q(\theta')) + m(\theta') + \theta' - \theta'' \\ &> V(Q^\epsilon(\theta')) + m^\epsilon(\theta') + \theta' - \theta'', \end{aligned}$$

which implies that

$$\begin{aligned} & V(Q(\theta'')) + m(\theta'') - V(Q(f^\epsilon(\theta'))) - m(f^\epsilon(\theta')) - \theta' \\ = & \int_{f^\epsilon(\theta')}^{\theta''} V_Q(Q(x))Q_\theta(x) + m_\theta(x) + 1\, dx > 0. \end{aligned}$$

By the construction of f^ϵ, we know that $f^\epsilon(\theta') \in [\theta_1, \theta_2] \subset B$, and the same argument as above yields our contradiction. Therefore, $(Q^\epsilon, m^\epsilon + \theta)$ is incentive compatible, so the principal's expected utility is

$$\begin{aligned} & \int_{\underline{\theta}}^{\theta_1} [U(Q(x)) - C(Q(x)) - m(x) - x] f(x)\, dx \\ + & \int_{\theta_1}^{\theta_2 - \epsilon} [U(Q(\theta_2)) - C(Q(\theta_2)) - m(\theta_2) - \theta_2] f(x)\, dx \\ + & \int_{\theta_2 - \epsilon}^{\theta_2} [U(Q^\epsilon(x)) - C(Q^\epsilon(x)) - m^\epsilon(x) - x] f(x)\, dx \\ + & \int_{\theta_2}^{\overline{\theta}} [U(Q(x)) - C(Q(x)) - m(x) - x] f(x)\, dx. \end{aligned}$$

Note that the principal's expected payoff is a continuous function of ϵ and that it converges to his payoff under $(Q', m' + \theta)$ as ϵ goes to zero. Then we can find an ϵ small enough to make his payoff arbitrarily close to his payoff under $(Q', m'+\theta)$, and in particular, we can find an ϵ such that $(Q^\epsilon, m^\epsilon+\epsilon)$ does strictly better for the principal than the original mechanism $(Q, m+\theta)$.

Now consider (**II**). First, we need to establish that $Q_\theta \leq 0$ implies $m_\theta + 1 \geq 0$. This follows easily from the Kuhn-Tucker conditions for the agent's problem:

$$m_\theta + 1 = -\frac{V_Q Q_\theta}{1+\mu} \geq 0,$$

where we use the fact that $\mu \geq 0$. Next, note that a consequence of $m_\theta+1 \geq 0$ is $V_Q Q_\theta + m_\theta + 1 \leq 0$. To see this, recall that the Kuhn-Tucker condition for the agent's problem

$$V_Q Q_\theta + m_\theta + 1 + \mu(m_\theta + 1) = 0$$

implies that

$$V_Q Q_\theta + m_\theta + 1 = -\mu(m_\theta + 1) \leq 0.$$

Therefore, to prove the second part of the proposition, we just have to rule out the possibility that $V_Q Q_\theta + m_\theta + 1 < 0$ for an optimal incentive compatible mechanism. Since $m_\theta + 1 \geq 0$, we have two cases: $m_\theta + 1 = 0$ and $m_\theta + 1 > 0$. In the first case, $V_Q Q_\theta + m_\theta + 1 < 0$ and $V_Q > 0$ imply that $Q_\theta = 0$, so we have the desired condition $V_Q Q_\theta + m_\theta + 1 = 0$.

In the second case, depicted in Figure 8 (top), suppose in order to show a contradiction that $\exists\theta' \in \Theta$ such that $V_Q(Q(\theta'))Q_\theta(\theta') + m_\theta(\theta') + 1 < 0$ and $m_\theta(\theta') + 1 > 0$.

[Figure 8 about here.]

Note that since the left hand side in the former inequality is a continuous function of θ, $\exists\delta' > 0$ such that $\theta \in B_{\delta'}(\theta')$ implies $V_Q(Q(\theta))Q_\theta(\theta) + m_\theta(\theta) + 1 < 0$ And since $m_\theta + 1$ is a continuous function of θ, we know $\exists\delta'' > 0$ such that $\theta \in B_{\delta''}(\theta')$ implies $m_\theta(\theta) + 1 > 0$. Let $\delta = \min\{\delta', \delta''\}$. Therefore, $\forall\theta \in B_\delta(\theta')$

$$V_Q Q_\theta + m_\theta + 1 < 0$$

and

$$m_\theta + 1 > 0.$$

From these inequalities and the Kuhn-Tucker condition for the agent's problem we know $\forall\theta \in B_\delta(\theta')$ $V_Q Q_\theta + m_\theta + 1 = -\mu(m_\theta + 1) < 0$, which implies $\mu > 0$. We then have $\forall\theta \in B_\delta(\theta')$

$$Q_\theta < 0$$

and

$$m(\theta) = 0,$$

where the last equation follows from the complementary slackness condition.

Note that the quasi-linearity of the principal's utility function implies that the first best technology level for the principal $Q^*(\theta)$ is a constant Q^*, and furthermore, Q^* is unique since $U_{QQ} - C_{QQ} < 0$. Next, we need to show that there is an open set $B \subset B_\delta(\theta')$ the image of which lies either entirely above Q^* or entirely below it. If this is the case for $B_\delta(\theta')$ itself then we have our set. Suppose, on the other hand, that $\exists\theta_1, \theta_2 \in B_\delta(\theta')$ such that $\theta_1 < \theta_2$ and $Q(\theta_1) \geq Q^*$ and $Q(\theta_2) \leq Q^*$. Then since $\forall\theta \in B_\delta(\theta')\ Q_\theta(\theta) < 0$, the open set $B = B_\delta(\theta') \cap (-\infty, \theta_1)$ satisfies our purposes. Therefore, without loss of generality we take an open set $B \subseteq B_\delta(\theta')$ such that $\forall\theta \in B\ Q(\theta) > Q^*$.

Picking two arbitrary points $\theta_1, \theta_2 \in B$ with $\theta_1 < \theta_2$, we will alter the original mechanism $\theta \mapsto (Q(\theta), m(\theta) + \theta)$ between these points by subtracting a variation ν from the value of $Q(\theta)$, where the function ν satisfies certain conditions to be specified. First, let

$$d = \min_{\theta\in[\theta_1,\theta_2]} -\left(\frac{Q_\theta(\theta)}{m_\theta(\theta)+1} + \frac{1}{V_Q(Q(\theta))}\right),$$

be the minimum vertical distance between $Q_\theta/(m_\theta + 1)$ and $-1/V_Q$ over the interval $[\theta_1, \theta_2]$, where d is well defined since we are minimizing a continuous function over a compact set. Furthermore, since $[\theta_1, \theta_2] \subset B \subset B_\delta(\theta')$, we know that the right hand side is positive over the entire interval, and since the right hand side is a continuous function it must assume its minimum d at some point in $[\theta_1, \theta_2]$. It follows that $d > 0$. Now, pick a variation $\nu : [\theta_1, \theta_2] \to \Re_+$ such that ν is continuously differentiable, and

$$\begin{aligned} 0 &= \nu(\theta_1) = \nu(\theta_2) \\ 0 &> |\nu_\theta| - d/2 \\ 0 &> |\nu_\theta| + Q_\theta \end{aligned}$$

and $\forall\theta \in (\theta_1, \theta_2)\ \nu(\theta) > 0$. Roughly, ν must be non-negative and sufficiently flat, but positive over the open interval. That the third condition can be satisfied by some function ν follows from the fact that $\forall\theta \in B \subset B_\delta(\theta')\ Q_\theta(\theta) < 0$.

Altering the original mechanism by subtracting ν, we get the new mechanism $\theta \mapsto (Q'(\theta), m'(\theta)+\theta)$ defined by

$$(Q'(\theta), m'(\theta) + \theta) = \begin{cases} (Q(\theta), m(\theta) + \theta) & \text{if } \theta \leq \theta_1 \\ (Q(\theta) - \nu(\theta), m(\theta) + \theta) & \text{if} \theta_1 \leq \theta \leq \theta_2 \\ (Q(\theta), m(\theta) + \theta) & \text{if } \theta_2 \leq \theta, \end{cases}$$

as depicted in Figure 8 (bottom). Note that this new mechanism is a member of the class of piecewise continuously differentiable mechanisms, so we just have to check that the utility of the principal is greater under $(Q', m' + \theta)$ than $(Q, m + \theta)$. To do this, it will be useful to verify that the new mechanism is incentive compatible, and by Proposition 11 it suffices to check that only four conditions hold for $(Q', m' + \theta)$:

$$\begin{aligned} 0 &\geq Q'_\theta \\ 0 &\leq m'_\theta + 1 \\ 0 &\geq V_Q Q'_\theta + m'_\theta + 1 \\ 0 &= m'(\theta)(V_Q Q'_\theta + m'_\theta + 1). \end{aligned}$$

The second inequality is obviously fulfilled, since $m' \equiv m$ and $m_\theta + 1 \geq 0$. To see that the fourth equation holds, note that $\forall \theta \in B \subset B_\delta(\theta')$ $m'(\theta) = m(\theta) = 0$.

To verify that $Q'_\theta \leq 0$, note that (I) implies this is true for the set $\Theta \setminus [\theta_1, \theta_2]$, so it remains only to be shown that it is true on the interval $[\theta_1, \theta_2]$. By the conditions for ν we have $\forall \theta \in [\theta_1, \theta_2]$

$$Q'_\theta(\theta) = Q_\theta(\theta) - \nu_\theta(\theta) \leq Q_\theta(\theta) + |\nu_\theta(\theta)| < 0,$$

which gives us the first inequality. Again, the third inequality holds for the original mechanism $(Q, m + \theta)$, so it remains only to be shown that it is true on the interval $[\theta_1, \theta_2]$. Since $\forall \theta \in [\theta_1, \theta_2]$ $m'(\theta) = 0$, it follows that $m'_\theta(\theta) = 0$, so the condition can be rewritten as

$$Q'_\theta(\theta) = Q_\theta(\theta) - \nu_\theta(\theta) \leq -\frac{1}{V_Q(Q(\theta))}.$$

Also, since $m'_\theta = 0$ on $[\theta_1, \theta_2]$, we know by the construction of d that $\forall \theta \in [\theta_1, \theta_2]$

$$-Q_\theta(\theta) - \frac{1}{V_Q(Q(\theta))} \geq d > \frac{d}{2} > 0,$$

which implies

$$Q_\theta(\theta) + \frac{d}{2} < -\frac{1}{V_Q(\theta)}.$$

Then since ν is assumed to satisfy $|\nu_\theta| < d/2$, we know $Q_\theta + |\nu_\theta| < -1/V_Q$, yielding $\forall \theta \in [\theta_1, \theta_2]$

$$Q'_\theta(\theta) = Q_\theta(\theta) - \nu_\theta(\theta) \leq Q_\theta(\theta) + |\nu_\theta(\theta)| < -\frac{1}{V_Q(Q(\theta))},$$

which is the desired inequality. Therefore, the new mechanism $(Q', m' + \theta)$ is incentive compatible.

The last step is to show that the principal's expected utility is higher under the new mechanism than under the old. Under $(Q', m' + \theta)$, the principal's expected utility is just

$$\begin{aligned} &\int_{\underline{\theta}}^{\theta_1} [U(Q(x)) - C(Q(x)) - m(x) - x] f(x)\, dx \\ &+ \int_{\theta_1}^{\theta_2} [U(Q(x) - \nu(x)) - C(Q(x) - \nu(x)) - x] f(x) dx \\ &+ \int_{\theta_2}^{\overline{\theta}} [U(Q(x)) - C(Q(x)) - m(x) - x] f(x)\, dx, \end{aligned}$$

where the fact is used that $m = 0$ on the interval $[\theta_1, \theta_2]$. It will therefore suffice to show that

$$\int_{\theta_1}^{\theta_2} [U(Q(x) - \nu(x)) - U(Q(x)) - C(Q(x) - \nu(x)) + C(Q(x))] f(x)\, dx > 0.$$

The result will follow if it can be shown that the integrand is positive over the interval (θ_1, θ_2).

Pursuing this line of argument, pick an arbitrary $\hat{\theta} \in (\theta_1, \theta_2)$. Then the integrand is

$$U(\hat{Q} - \hat{\nu}) - U(\hat{Q}) - C(\hat{Q} - \hat{\nu}) + C(\hat{Q}),$$

where $\hat{Q} = Q(\hat{\theta})$ and $\hat{\nu} = \nu(\hat{\theta})$. Recalling that $\forall \theta \in (\theta_1, \theta_2)\ \nu(\theta) > 0$, this can be rewritten as

$$-\int_0^{\hat{\nu}} U_Q(\hat{Q} - x) - C_Q(\hat{Q} - x)\, dx,$$

so once again the result will follow if it can be shown that the integrand is negative over the interval $[0, \hat{\nu}]$. Note from the first order conditions which characterize Q^* and the assumptions on U and C that $\forall Q > Q^*\ U_Q(Q) - C_Q(Q) < 0$, so we have the result if it can be shown that $\forall x \in [0, \hat{\nu}]\ \hat{Q} - x > Q^*$. Furthermore, if this is true for $x = \hat{\nu}$ then it is true for all points in the interval. Using the assumption that $\nu(\theta_2) = 0$, note that

$$\hat{Q} - \hat{\nu} = Q(\hat{\theta}) - \nu(\hat{\theta}) = Q(\theta_2) - \int_{\hat{\theta}}^{\theta_2} Q_\theta(x) - \nu_\theta(x)\, dx.$$

Since $\forall \theta \in [\theta_1, \theta_2]\ Q_\theta'(\theta) = Q_\theta(\theta) - \nu_\theta(\theta) < 0$, it follows that

$$\int_{\hat{\theta}}^{\theta_2} Q_\theta(x) - \nu_\theta(x)\, dx < 0,$$

so $\hat{Q} - \hat{\nu} > Q(\theta_2)$. Noting that $\theta_2 \in B$ implies $Q(\theta_2) > Q^*$, we have

$$Q(\hat{\theta}) - \nu(\hat{\theta}) > Q(\theta_2) > Q^*,$$

which gives us our contradiction. □

Proposition 14 *If the utility functions of the principal and agent are quasi-linear, the optimal incentive compatible mechanism $\theta \mapsto (Q(\theta), m(\theta) + \theta)$ is given by*

$$(Q(\theta), m(\theta) + \theta) = \begin{cases} (V^{-1}(V(Q(\overline{\theta})) + \overline{\theta} - \theta), \theta) & \text{if } \theta \geq \theta^* \\ (\overline{Q}, \theta^*) & \text{if } \theta \leq \theta^*. \end{cases}$$

Proof: Let $\Theta_1 = \{\theta \in \Theta | Q(\theta) < \overline{Q}\}$ and $\Theta_2 = \{\theta \in \Theta | Q(\theta) = \overline{Q}\}$, and note from Lemmata 4 and 5 that $\Theta = \Theta_1 \cup \Theta_2$. First, consider $\{\theta \in \Theta | \theta \geq \theta^*\} = \Theta_1 \cup \{\theta^*\}$. We know from the proof of Lemma 6 that $\forall \theta \in \Theta_1\ m_\theta(\theta) = -V_Q(Q(\theta))Q_\theta(\theta) - 1$, and since θ^* is an accumulation point of Θ_1 and m_θ, V_Q, Q, and Q_θ are continuous it follows that this is true for θ^* as well. Rewriting the equation, we have

$$m_\theta(\theta) + 1 = -V_Q(Q(\theta))Q_\theta(\theta),$$

and integrating yields

$$\begin{aligned} m(\theta) + \theta &= m(\overline{\theta}) + \overline{\theta} + \int_\theta^{\overline{\theta}} V_Q(Q(x))Q_\theta(x)\, dx \\ &= m(\overline{\theta}) + \overline{\theta} + V(Q(\overline{\theta})) - V(Q(\theta)). \end{aligned}$$

By Lemma 4, we know $\forall\theta \in \Theta_1$ $m(\theta) = 0$, and since θ^* is an accumulation point of Θ_1 and m is continuous it follows that this is true for θ^* as well. Then the above equation reduces to

$$\theta = \overline{\theta} + V(Q(\overline{\theta})) - V(Q(\theta)),$$

which can be rewritten as

$$Q(\theta) = V^{-1}(V(Q(\overline{\theta})) + \overline{\theta} - \theta).$$

Since $m(\theta) = 0$ implies $m(\theta) + \theta = \theta$, we have the desired result.

Now consider $\{\theta \in \Theta | \theta \leq \theta^*\} = \Theta_2$. Lemma 5 shows that $\forall\theta \in \Theta$ $Q(\theta) \leq \overline{Q}$, so from $Q_\theta \leq 0$ it follows that $\forall\theta \in \Theta_2$ $Q(\theta) = \overline{Q}$. It then remains only to be shown that $m(\theta) + \theta = \theta^*$. Note from the first part of the proof that $m(\theta^*) + \theta^* = \theta^*$, so the result will follow if it can be shown that $m(\theta) + \theta$ is constant on Θ_2. To see that this is the case, note that

$$m_\theta(\theta) + 1 = -V(Q(\theta))Q_\theta(\theta) = 0,$$

since $Q_\theta(\theta) = 0$ on Θ_2. □

Proposition 15 *If the utility functions of the principal and agent are quasi-linear then the incentive compatible mechanism $\theta \mapsto (Q(\theta), m(\theta) + \theta)$ is optimal only if $Q(\overline{\theta}) \leq Q^*$.*

Proof: Suppose in order to show a contradiction that $(Q(\theta), m(\theta) + \theta)$ is an optimal incentive compatible mechanism with endpoint $Q(\overline{\theta}) > Q^*$, and take the incentive compatible mechanism $(Q', m' + \theta)$ constructed as in Proposition 14 with endpoint $Q'(\overline{\theta}) = Q^*$. Let θ_1^* be uniquely defined as in Proposition 14 by

$$\overline{\theta} + V(Q(\overline{\theta})) - \theta_1^* \equiv V(\overline{Q}),$$

and note that θ_1^* can be considered an implicit function of $Q(\overline{\theta})$ with

$$\frac{d}{dQ(\overline{\theta})}\theta_1^* = V_Q(Q(\overline{\theta})) > 0.$$

Recalling that $Q(\overline{\theta}) > Q^*$, define θ_2^* by

$$\overline{\theta} + V(Q^*) - \theta_2^* \equiv V(\overline{Q}),$$

and note that $\theta_2^* < \theta_1^*$, as shown in Figure 9.

[Figure 9 about here.]

Since $(Q', m' + \theta)$ and $(Q, m + \theta)$ are both given by the construction in Proposition 14, the principal does better under the new mechanism if the following condition holds:

$$\begin{aligned}
&\int_{\underline{\theta}}^{\theta_2^*} (\theta_1^* - \theta_2^*) f(x)\, dx \\
&+ \int_{\theta_2^*}^{\theta_1^*} [U(Q'(x)) - U(\overline{Q}) - C(Q'(x)) + C(\overline{Q}) - x + \theta_1^*] f(x)\, dx \\
&+ \int_{\theta_1^*}^{\overline{\theta}} [U(Q'(x)) - U(Q(x)) - C(Q'(x)) + C(Q(x))] f(x)\, dx > 0.
\end{aligned}$$

The first term is obviously positive, since $\theta_1^* > \theta_2^*$. To see that the second term is positive, pick an arbitrary $x \in (\theta_2^*, \theta_1^*)$. The endpoints of the integral give $\theta_1^* - x > 0$, and we also know that

$$\begin{aligned} & U(Q'(x)) - U(\overline{Q}) - C(Q'(x)) + C(\overline{Q}) \\ = & \int_{\theta_2^*}^{x} U_Q(Q'(x))Q'_\theta(x) - C_Q(Q'(x))Q'_\theta(x)\, dx. \end{aligned}$$

Furthermore, we know from the definition of Q^* and our assumptions on U and C that $\forall Q > Q^*$ $U_Q(Q) - C_Q(Q) < 0$. By the construction in Proposition 14, we know $\theta_2^* < x < \overline{\theta}$ implies $Q'(x) > Q^*$ and $Q'_\theta(x) < 0$, it follows that

$$(U_Q(Q'(x)) - C_Q(Q'(x)))Q'_\theta(x) > 0,$$

which gives us the desired result.

A similar argument establishes that the third term is positive. First, picking an arbitrary $x \in (\theta_1^*, \overline{\theta})$, note from Proposition 14 that $Q(x) = V^{-1}(V(Q(\overline{\theta})) + \overline{\theta} - x)$ and $Q'(x) = V^{-1}(V(Q^*) + \overline{\theta} - x)$. Since the derivative with respect to $V(Q(\overline{\theta}))$ in the expression for Q is positive and $V(Q(\overline{\theta})) > V(Q^*)$, it follows that $Q'(x) < Q(x)$. Next, define $h = Q(x) - Q'(x) > 0$ and note that

$$\begin{aligned} & U(Q'(x)) - U(Q(x)) - C(Q'(x)) + C(Q(x)) \\ = & U(Q'(x)) - U(Q'(x) + h) - C(Q'(x)) + C(Q'(x) + h) \\ = & -\int_0^h U_Q(Q'(x) + y) - C_Q(Q'(x) + y)\, dy. \end{aligned}$$

By the construction in Proposition 14, we know that $Q'(x) > Q^*$, and consequently, $Q'(x) + y > Q^*$. Then $\forall Q > Q^*$ $U_Q(Q) - C_Q(Q) < 0$ implies that

$$-\int_0^h U_Q(Q'(x) + y) - C_Q(Q'(x) + y)\, dy > 0,$$

which is the desired result. □

Corollary 17 *If the utility functions of the principal and agent are quasi-linear then the optimal menu of fixed price contracts* $\{(Q^f(\theta, B^f), m^f(\theta, B^f) + \theta) \in \Re_+^2 | \theta \in \Theta\}$ *is suboptimal.*

Proof: In order to apply Proposition 14, it is first necessary to show that the optimal menu of fixed price contracts is incentive compatible. By Proposition 11, there are four properties to verify. It was shown in Section 3 that $Q_\theta^f \leq 0$ and that $m_\theta^f + 1 \geq 0$. To see that $m^f(\theta, B^f)(V_Q Q_\theta^f + m_\theta^f + 1) = 0$, suppose that $m^f(\theta, B^f) > 0$. It was shown in Section 3 that for the quasi-linear case $m^f(\theta, B^f) > 0$ implies $Q_\theta^f = m_\theta^f + 1 = 0$, so obviously $V_Q Q_\theta^f + m_\theta^f + 1 = 0$. Finally, to see that $V_Q Q_\theta^f + m_\theta^f + 1 \leq 0$, suppose that $m^f(\theta, B^f) = 0$. It was shown in Section 3 that $m^f(\theta, B^f) = 0$ implies $Q_\theta^f = -1/C_Q$ and $m_\theta^f = 0$. The Kuhn-Tucker conditions for the agent's fixed price problem are

$$V_Q - C_Q - \lambda C_Q = 0,$$

where $\lambda \geq 0$, $B^f - C(Q^f) - \theta \geq 0$, and $\lambda(B^f - C(Q^f) - \theta) = 0$. It follows that $V_Q = C_Q + \lambda C_Q \geq C_Q$, or in other words, $V_Q / C_Q \geq 1$. But then substituting for Q^f_θ and m^f_θ, we have

$$V_Q Q^f_\theta + m^f_\theta + 1 = -\frac{V_Q}{C_Q} + 1 \leq 0,$$

which is the desired result. Therefore, the optimal menu of fixed price contracts is indeed incentive compatible, and it must satisfy the conditions of Proposition 14.

To see that the fixed price menu of contracts is suboptimal, two cases must be considered: $\forall \theta \in [\underline{\theta}, \overline{\theta})$ $m^f(\theta, B^f) > 0$ and $\exists \theta' \in [\underline{\theta}, \overline{\theta})$ such that $m^f(\theta', B^f) = 0$. In the first case, it is shown in Section 3 that when the agents have quasi-linear utility functions $\forall \theta \in [\underline{\theta}, \overline{\theta})$ $Q^f_\theta = 0$ and $m^f_\theta + 1 = 0$. This implies that $\forall \theta \in \Theta$ $Q^f(\theta, B^f) = Q^f(\overline{\theta}, B^f)$ and $m^f(\theta, B^f) + \theta = \overline{\theta}$, since

$$Q^f(\theta, B^f) = Q^f(\overline{\theta}, B^f) - \int_{[\theta, \overline{\theta})} Q^f_\theta(x, B^f)\, dx = Q^f(\overline{\theta}, B^f)$$

and $m^f_\theta = -1$ implies that

$$m^f(\theta, B^f) = m^f(\overline{\theta}, B^f) + \int_{[\theta, \overline{\theta})} dx = \overline{\theta} - \theta.$$

But this is just a cost-plus contract $(Q^f(\overline{\theta}, B^f), \overline{\theta})$, which was shown to be suboptimal in Corollary 16.

In the second case, suppose $\exists \theta' \in [\underline{\theta}, \overline{\theta})$ such that $m^f(\theta', B^f) = 0$. By the construction in Proposition 14 it follows that $\forall \theta \in [\theta', \overline{\theta}]$ $m^f(\theta, B^f) = 0$. Then $\forall \theta \in (\theta', \overline{\theta})$

$$\mathcal{V}_\theta(\theta') = V_Q(Q^f(\theta'))Q^f_\theta(\theta') - C_Q(Q^f(\theta'))Q^f_\theta(\theta') - 1 = -\frac{V_Q(Q^f(\theta'))}{C_Q(Q^f(\theta'))},$$

where $Q^f_\theta = -1/C_Q$. But by Proposition 14, $\theta' \in \Theta_1 \cup \{\theta^*\}$ and if the fixed price menu of contracts is optimal we must have

$$\mathcal{V}(\theta') = V(Q(\overline{\theta})) + \overline{\theta} - \theta'$$

and

$$\mathcal{V}_\theta(\theta') = -1 \neq -\frac{V_Q(Q^f(\theta'))}{C_Q(Q^f(\theta'))},$$

where the failure of equality is generally true since V is strictly concave and C is strictly convex. That is, the utility of agent types decreases at a rate inconsistent with the construction in Proposition 14, and therefore, the optimal menu of fixed price contracts is suboptimal. □

B Lemmata

Lemma 1 *A type θ_2 agent can successfully imitate a type $\theta_1 < \theta_2$ agent if and only if $m(\theta_1) + \theta_1 \geq \theta_2$.*

Proof: First, consider the "only if" direction, and take type θ_1 and θ_2 agents such that $\theta_1 < \theta_2$ and $m(\theta_1) + \theta_1 < \theta_2$, and suppose the type θ_2 agent attempts to imitate the type θ_1 agent. Since the principal observes the quality level, the type θ_2 agent must produce $Q(\theta_1)$, but even if he picks $m = 0$ his cost will be $C(Q(\theta_1))+\theta_2 > C(Q(\theta_1))+\theta_1$. Since the principal also observes the total cost of the project, he will sue the agent for breach of contract, inflicting infinitely large costs on the type θ_2 agent.

The "if" direction follows easily, since under the antecedent assumption the type θ_2 agent can pick a level of misallocated funds m equal to $m(\theta_1) + \theta_1 - \theta_2 \geq 0$. □

Lemma 2 *If $Q_\theta \leq 0$ and $m_\theta + 1 \geq 0$, then $\mathcal{Q}$ is non-increasing.*

Proof: Suppose in order to show a contradiction that for $x', x'' \in M = [m(\underline{\theta} + \underline{\theta}, m(\overline{\theta}) + \overline{\theta}]$ with $x' < x''$ we have $\mathcal{Q}(x') < \mathcal{Q}(x'')$. We know $\exists \theta', \theta'' \in \Theta$ such that $m(\theta') + \theta' = x'$ and $m(\theta'') + \theta'' = x''$, and since $m_\theta + 1 \geq 0$ it must be the case that $\theta' < \theta''$. Now we have by assumption $\mathcal{Q}(x') = Q(\theta') < Q(\theta'') = \mathcal{Q}(x'')$. But since $Q_\theta \leq 0$ and $\theta' < \theta''$, we know that $Q(\theta'') \leq Q(\theta')$, a contradiction. □

Lemma 3 *If $Q_\theta \leq 0$, $m_\theta + 1 \geq 0$, and $m(\theta)(V_Q Q_\theta + V_m(m_\theta + 1)) = 0$ then $\mathcal{Q}$ is single-valued.*

Proof: First, note that $\mathcal{Q}$ cannot be sloping upward, as shown in the previous lemma. If $\mathcal{Q}$ is not single-valued, it then follows that $\exists \theta_1, \theta_2 \in \Theta, \theta_1 < \theta_2$ with $m(\theta_1) + \theta_1 = m(\theta_2) + \theta_2$ and $\mathcal{Q}(m(\theta_1) + \theta_1) > \mathcal{Q}(m(\theta_2) + \theta_2)$. Since $m_\theta + 1 \geq 0$, it must be the case that $\forall \theta \in [\theta_1, \theta_2]\ m(\theta) + \theta = m(\theta_1) + \theta_1 = m(\theta_2) + \theta_2$. That is, we only need to consider the possibility that $\mathcal{Q}$ is vertical at $m(\theta_1) + \theta_1$. To rewrite this yet another way, it will suffice to derive a contradiction from the assumption that $\forall \theta \in [\theta_1, \theta_2]\ m_\theta + 1 = 0$.

Note that $\forall \theta \in [\theta_1, \theta_2)\ m(\theta) > 0$, since $m(\theta) + \theta = m(\theta_2) + \theta_2$, which implies $m(\theta) = m(\theta_2) + \theta_2 - \theta > m(\theta_2) \geq 0$. By our assumptions, it follows that $\forall \theta \in [\theta_1, \theta_2)$

$$V_Q Q_\theta + V_m(m_\theta + 1) = V_Q Q_\theta = 0,$$

which implies that $Q_\theta = 0$, so Q is constant on the closed interval $[\theta_1, \theta_2]$. But then $Q(\theta_1) = Q(\theta_2)$, contradicting our assumption that $\mathcal{Q}(m(\theta_1) + \theta_1) = Q(\theta_1) < Q(\theta_2) = \mathcal{Q}(m(\theta_2) + \theta_2)$. □

Lemma 4 $Q(\theta) < \overline{Q}$ *implies* $m(\theta) = 0$.

Proof: We know that if $\exists \theta \in \Theta$ such that $Q(\theta) < \overline{Q}$ then $U_Q(Q(\theta)) - C_Q(Q(\theta)) + V_Q(Q(\theta)) > 0$. From (5) it then follows that $\gamma > 0$, which from (4) implies that $m = m^* = 0$. □

Lemma 5 $\forall \theta \in \Theta\ Q(\theta) \leq \overline{Q}$.

Proof: Suppose in order to show a contradiction that $\exists\theta \in \Theta$ such that $Q(\theta) > \overline{Q}$. Then $U_Q(Q(\theta)) - C_Q(Q(\theta)) + V_Q(Q(\theta)) < 0$, but then (5) implies $\gamma < 0$, which contradicts (4). □

Lemma 6 $\forall\theta \in \Theta\ Q_\theta(\theta) \leq 0$.

Proof: First, define $\Theta_1 = \{\theta \in \Theta | Q(\theta) < \overline{Q}\}$ and $\Theta_2 = \{\theta \in \Theta | Q(\theta) = \overline{Q}\}$, and note that $\Theta_2 = \Theta \setminus \Theta_1$. Since Lemma 4 implies that $\forall\theta \in \Theta_1\ m(\theta) = 0$, it follows that

$$\begin{aligned} m_\theta &= m^*_\theta(Q(\theta), \mathcal{V}(\theta))Q_\theta(\theta) + m^*_{\mathcal{V}}(Q(\theta), \mathcal{V}(\theta))\mathcal{V}_\theta \\ &= -V_Q(Q(\theta))Q_\theta(\theta) - 1 = 0. \end{aligned}$$

Rewriting the last equation, we have

$$Q_\theta(\theta) = -\frac{1}{V_Q(Q(\theta))} < 0,$$

so the proposition holds for Θ_1. By construction of Θ_2, the level of quality is constant at $\overline{Q}$, so $\forall\theta \in \Theta_2\ Q_\theta(\theta) = 0$, which gives us the desired result. □

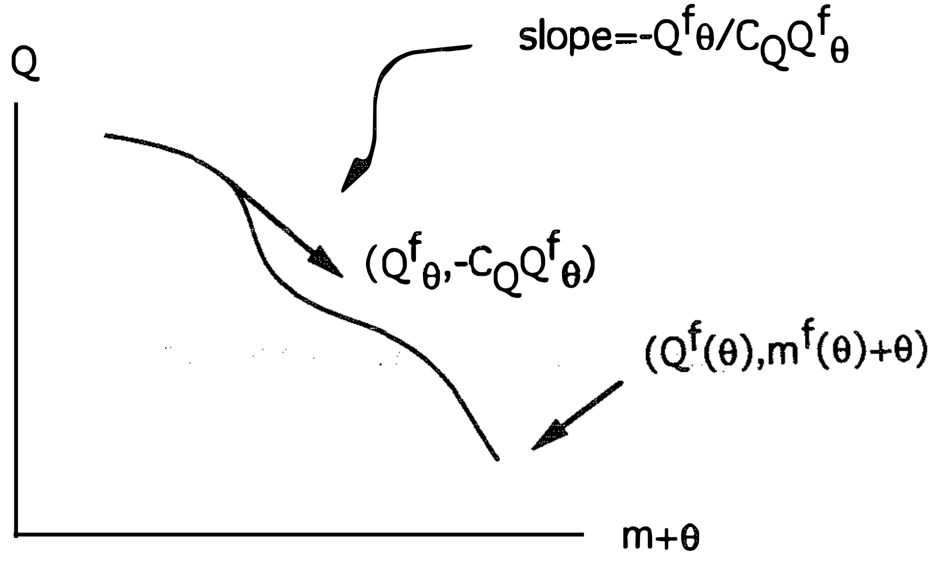


Figure 1: A menu of fixed price contracts

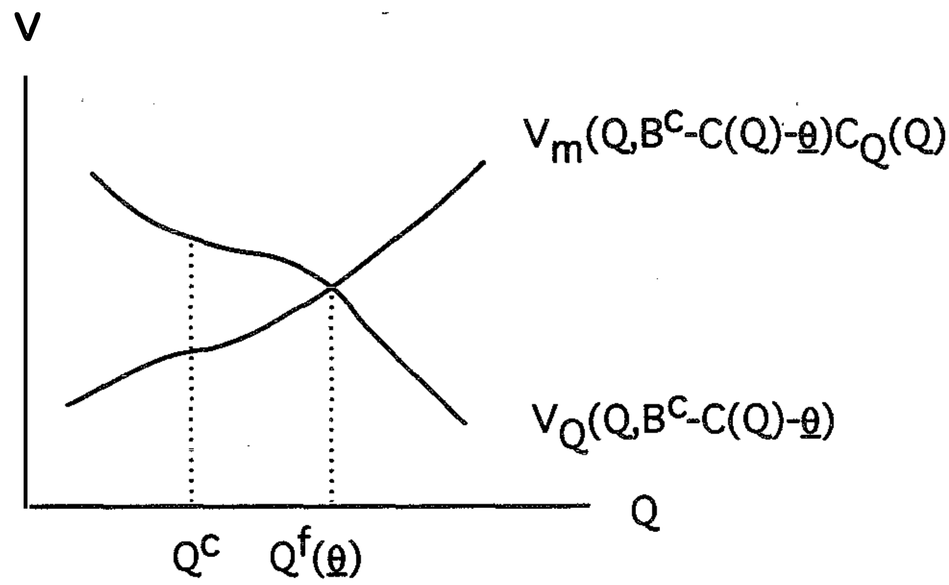


Figure 2: Agent's first order conditions

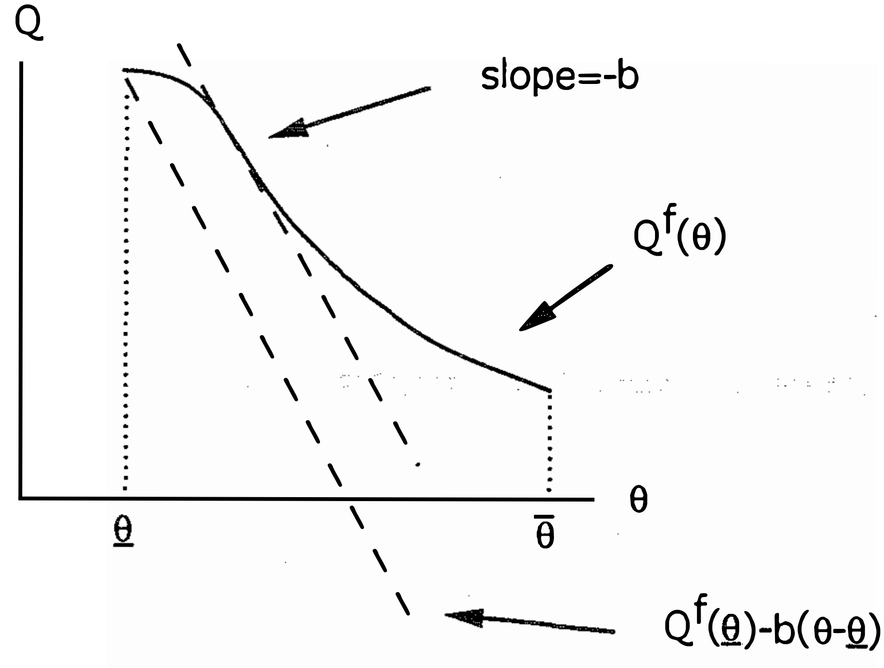


Figure 3: Quality is bounded below

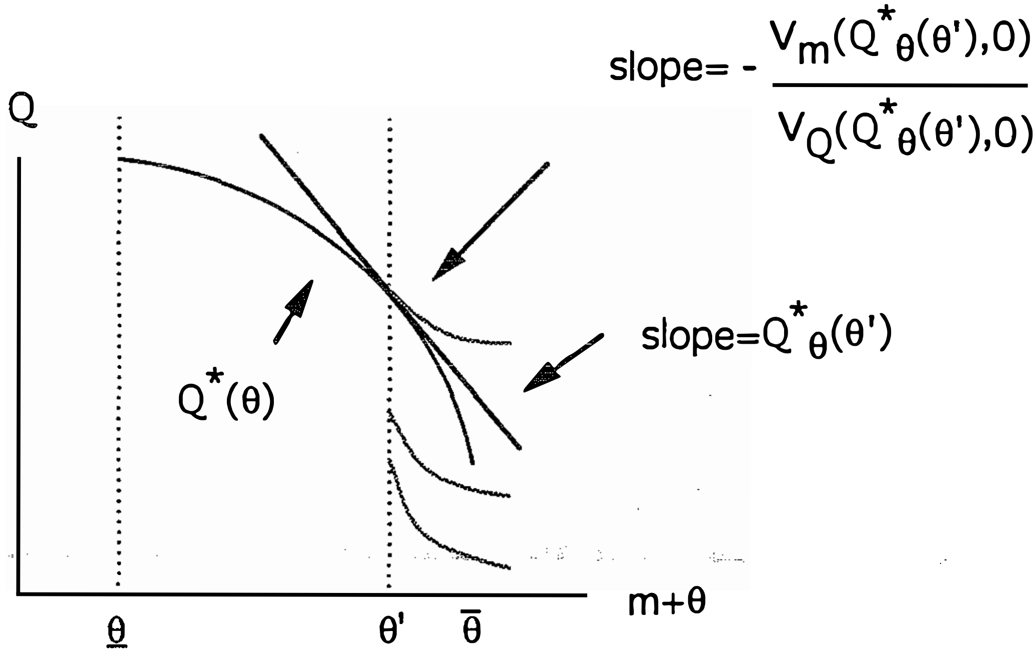


Figure 4: Curves in contract space

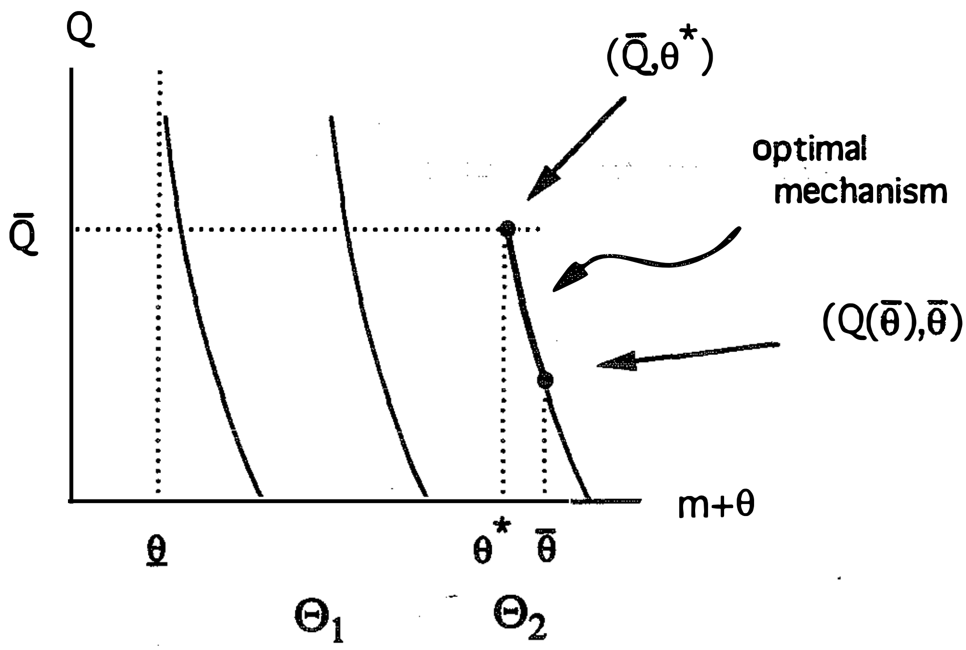
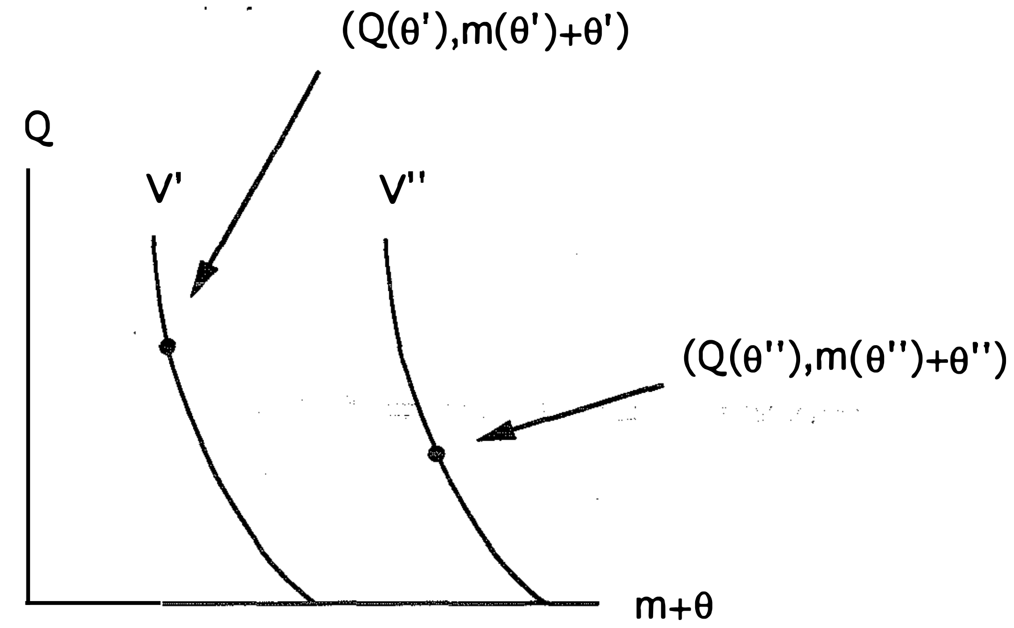


Figure 5: The optimal incentive compatible mechanism



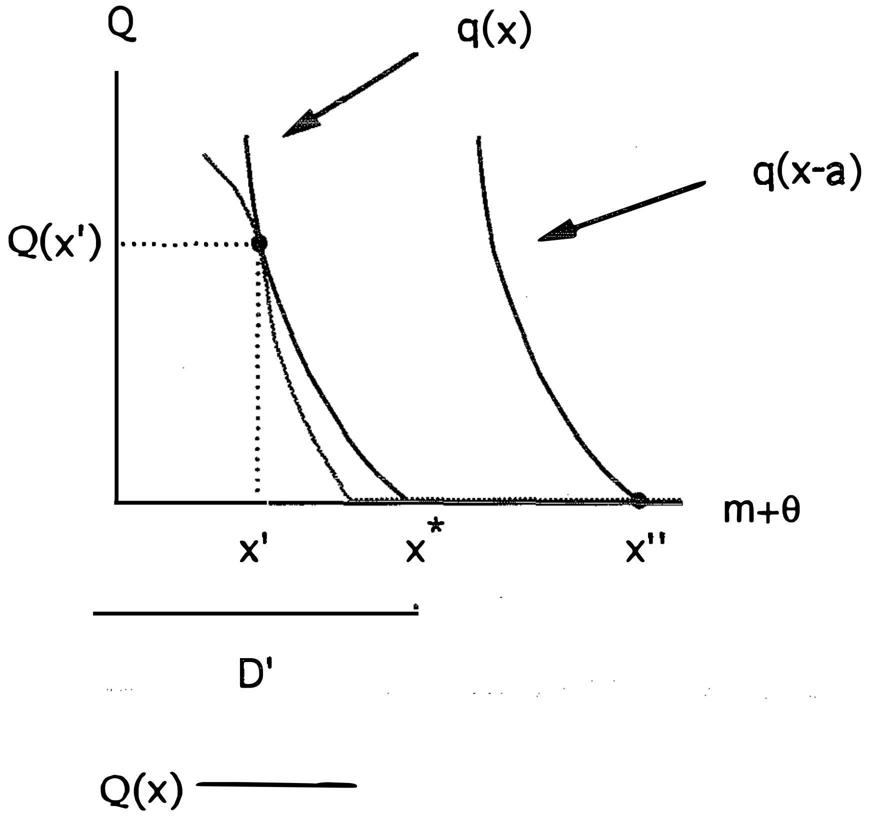
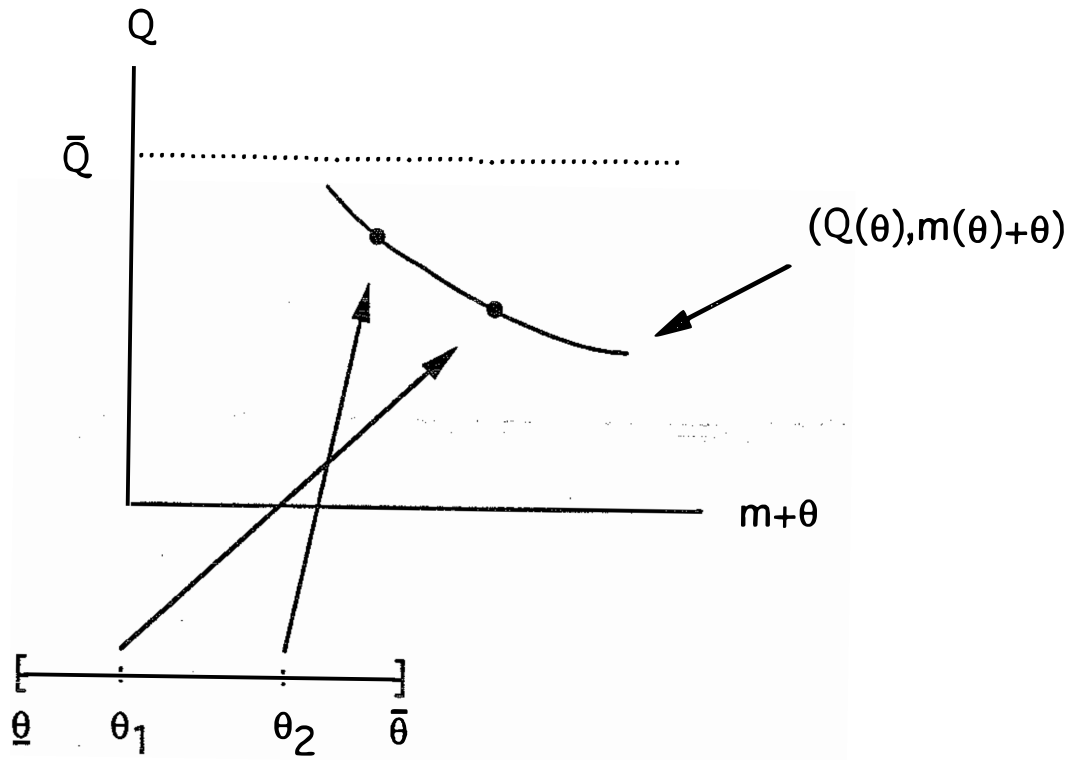


Figure 6: Proof of Proposition 11



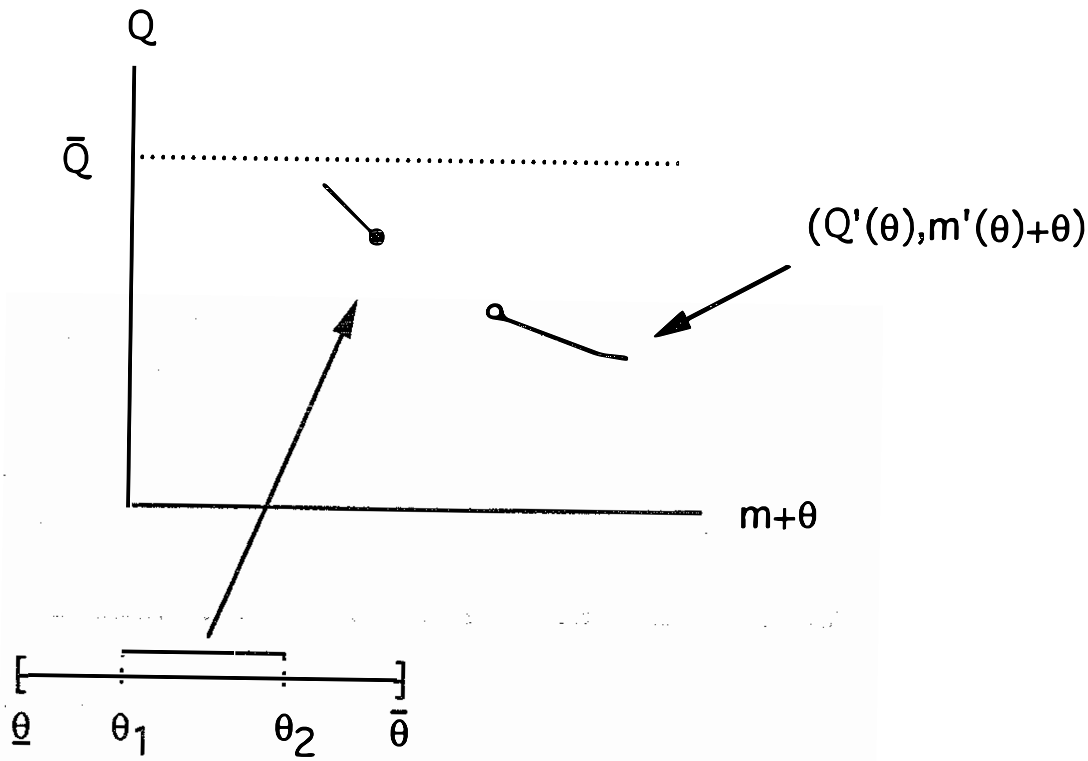


Figure 7: Proof of Proposition 12 **I**

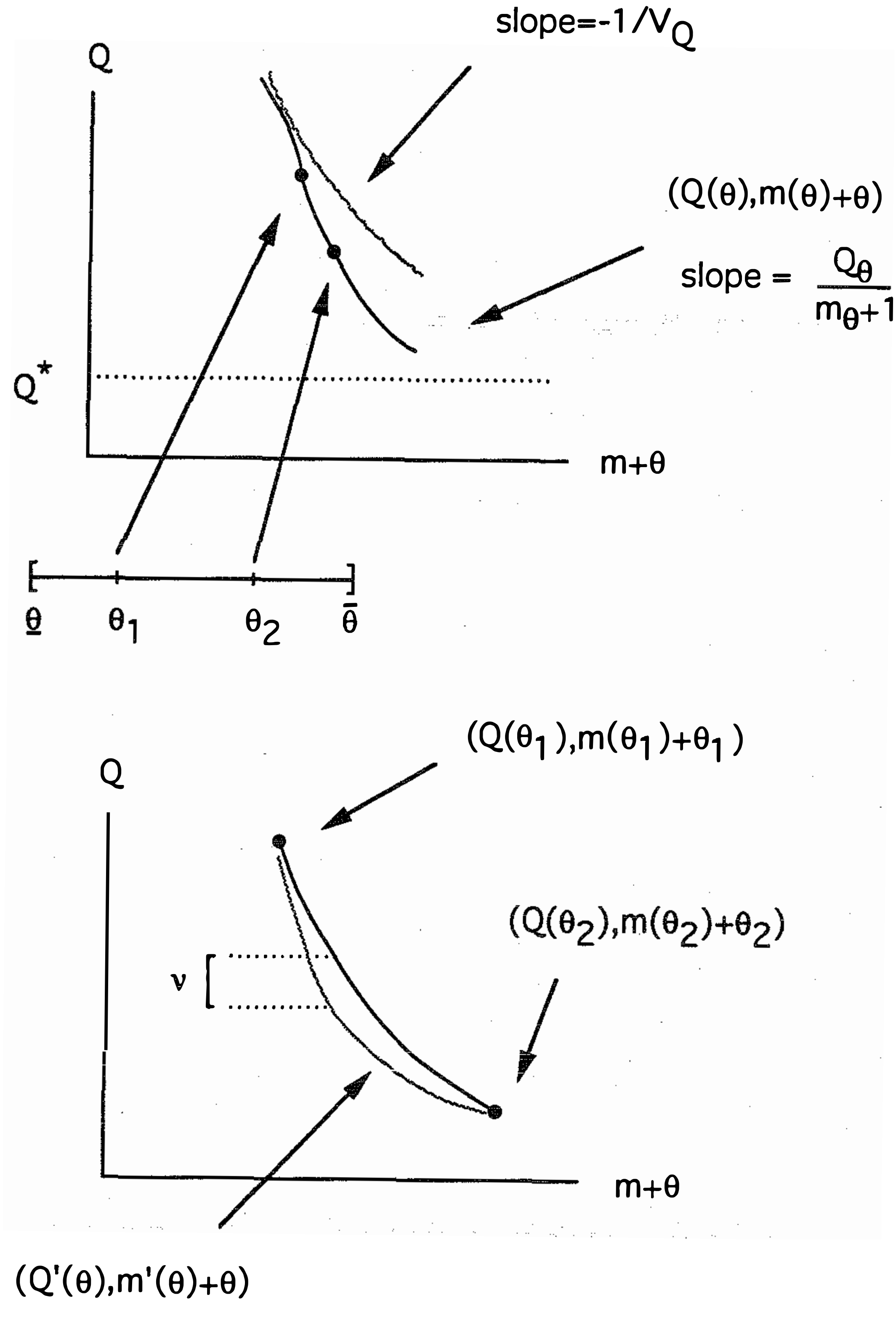


Figure 8: Proof of Proposition 12 **II**

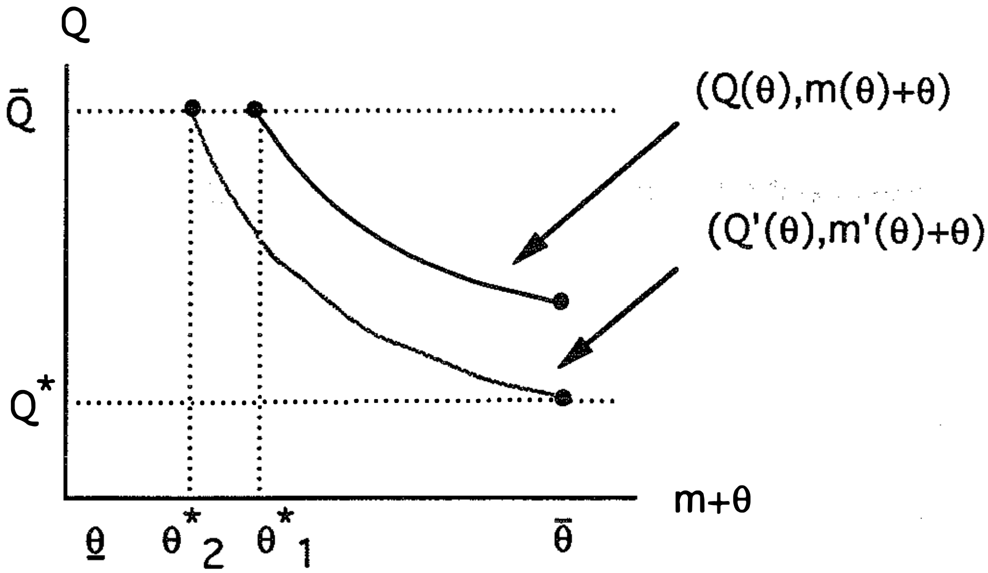


Figure 9: Proof of Proposition 15

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